The q-Eulerian distribution of the elliptic Weyl group of type $A_1^{(1,1)}$

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Abstract: We calculate the q-Eulerian distribution $W(t,q)$ of the elliptic Weyl group of type $A_1^{(1,1)}$, which is a formal power series in $\mathbf{Z}[[t,q]]$, and classically defined for any Coxeter system (W,S).

Key words: q-Eulerian distribution; elliptic Weyl group.

1. Introduction. Let (W,S) be a Coxeter system, then for $w \in W$, its length and descent number are defined by $l(w) = min\{l \colon w = s_{i_1}s_{i_2}\cdots s_{i_l}$ for some $s_{i_k} \in S\}$, $\operatorname{des}(w) = |\{s \in S \colon l(ws) < l(w)\}|$, and the bivariate generating function which is called the q-Eulerian distribution; $W(t,q) = \sum_{w\in W} t^{\operatorname{des}(w)}q^{l(w)}$ is defined ([2]). For example, when W is the symmetric group S_n, the following result was given by Stanley [3]; $\sum_{n\geq 0} x^n/[n]_q \sum_{w\in S_n} t^{des(w)}q^{l(w)} = (1-t)\exp(x(1-t)\colon q)/(1-t\exp(x(1-t)\colon q))$ where $\exp(x\colon q) = \sum_{n\geq 0} x^n/[n]_q$, $[n]_q = (1-q^n)/(1-q)$. When W is finite, $W(t,1)$ is called the Eulerian polynomial of W. In this paper, we calculate $W(t,q)$ for the elliptic Weyl group of type $A_1^{(1,1)}$ with the given generator system. Here we note that the elliptic Weyl groups are not Coxeter groups, but in a sense, generalized Coxeter groups ([1]), and we can define their lengths, descent numbers and q-Eulerian distribution similarly to Coxeter groups. In the case of the elliptic Weyl group $W\big(A_1^{(1,1)}\big)$ with the given generator system, the length distribution, which are also called Poincaré series; $W(q) = \sum_{w\in W} q^{l(w)}$ was calculated by Wakimoto [4], and in a different way by the author [5]. To calculate $W(t,q)$ for that, we use the previous result [5].

2. The q-Eulerian distribution of the elliptic Weyl group $W\big(A_1^{(1,1)}\big)$. The elliptic Weyl group $W\big(A_1^{(1,1)}\big)$ of type $A_1^{(1,1)}$ is presented as follows ([1]):

Generators: w_i, w_i^* $(i = 0,1)$.

Relations: $w_i^2 = w_i^{*2} = 1$ $(i = 0,1)$,

$$w_0w_0^*w_1w_1^* = 1.$$

We set $T = w_1w_0$, $R = w_1^*w_1 = w_0w_0^*$, then there hold the relations; $TR = RT$, $w_1T = T^{-1}w_1$, $w_1R = R^{-1}w_1$. We calculate $W(t,q)$ by using the same method as [5]. Noting $W\big(A_1^{(1,1)}\big) = \{R^mT^nw_1, R^mT^n, m,n \in \mathbf{Z}\}$, we divide them into the following cases; {(I) T^n $(n\geq 0)$, (II) T^{-n} $(n\geq 1)$, (III) T^nw_1 $(n\geq 0)$, (IV) $T^{-n}w_1$ $(n\geq 1)$}, and multiply the elements R^m $(m\in\mathbf{Z})$ on the left by those. Further we use the following

Lemma 2.1. (i) *Let w be a minimal expression by w_0 and w_1. Then even if we attach $*$ to any letters of w, the length of w does not decrease.*

(ii) *For a positive integer m,*

$$\begin{aligned} R^m = (w_1^*w_1)^m &= \underbrace{w_1^*w_0^*w_1^*\cdots w_k^*}_{m}\ \underbrace{w_k\cdots w_1w_0w_1}_{m} \\ = (w_0w_0^*)^m &= \underbrace{w_0w_1w_0\cdots w_k}_{m}\ \underbrace{w_k^*\cdots w_0^*w_1^*w_0^*}_{m} \end{aligned}$$

where $w_k \in \{w_0,w_1\}$.

Proof. (i) is the same as [5], and (ii) is directly calculated. □

(I) $T^n = (w_1w_0)^n$ $(n\geq 0)$.

When $n = 0$, for $w = R^m = (w_1^*w_1)^m = (w_0w_0^*)^m$ $(m\geq 1)$, $\operatorname{des}(w) = |\{w_1, w_0^*\}| = 2$ and for $w = R^{-m} = (w_1w_1^*) = (w_0^*w_0)^m$ $(m\geq 1)$, $\operatorname{des}(w) = |\{w_1^*, w_0\}| = 2$.

When $n\geq 1$, we consider $w = R^kT^n$ $(k\geq 0)$. Noting the relation $Rw_1w_0 = w_1^*w_0 = w_1w_0^*$, and using Lemma 2.1 (i),

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$$\begin{cases} \text{(i)} & \text{if } k=0, \text{ then } w=(w_1w_0)^n, \text{ and} \\ & \operatorname{des}(w)=|\{w_0\}|=1, \\ \text{(ii)} & \text{if } 1\le k\le 2n-1, \text{ then} \\ & w=(w_{11}w_{10})\cdots(w_{n-1,1}w_{n-1,0})(w_{n,1}w_{n,0}), \\ & \text{where } w_{i1}\in\{w_1,w_1^*\}, w_{i0}\in\{w_0,w_0^*\}, \text{ and} \\ & \operatorname{des}(w)=|\{w_0,w_0^*\}|=2, \\ \text{(iii)} & \text{if } k=2n, \text{ then } w=(w_1^*w_0^*)^n, \text{ and} \\ & \operatorname{des}(w)=|\{w_0^*\}|=1. \end{cases}$$

Further using Lemma 2.1 (ii), for

$w=R^{2n+m}T^n=$

$$\begin{cases} =(w_0w_0^*)^m(w_1^*w_0^*)^n \\ =w_0w_1w_0\cdots w_kw_k^*\cdots w_0^*w_1^*w_0^*(w_1^*w_0^*)^n \\ =(w_1^*w_0^*)^n(w_1^*w_1)^m \\ =(w_1^*w_0^*)^nw_1^*w_0^*w_1^*\cdots w_k^*w_k\cdots w_1w_0w_1 \quad (m\ge 1), \end{cases}$$

$\operatorname{des}(w)=|\{w_0^*,w_1\}|=2$, and for

$w=R^{-m}T^n=$

$$\begin{cases} =(w_0^*w_0)^m(w_1w_0)^n \\ =w_0^*w_1^*w_0^*\cdots w_k^*w_k\cdots w_0w_1w_0(w_1w_0)^n \\ =(w_1w_0)^n(w_1w_1^*)^m \\ =(w_1w_0)^nw_1w_0w_1\cdots w_kw_k^*\cdots w_1^*w_0^*w_1^* \quad (m\ge 1), \end{cases}$$

$\operatorname{des}(w)=|\{w_0,w_1^*\}|=2$. From the above, we have;

$$W(t,q)_{(I)}=1+\sum_{m\ge1}2t^2q^{2m}+\sum_{n\ge1}2tq^{2n}+\sum_{n\ge1}(2n-1)t^2q^{2n}+\sum_{n\ge1,m\ge1}2t^2q^{2n+2m}.$$

From now on, similarly to (I) we calculate the others.

(II) $T^{-n}=(w_0w_1)^n$ $(n\ge1)$.

For $k\ge0$, we consider $w=R^{-k}T^{-n}$.

$$\begin{cases} \text{(i)} & \text{if } k=0, \text{ then } w=T^{-n}=(w_0w_1)^n, \text{ and} \\ & \operatorname{des}(w)=|\{w_1\}|=1, \\ \text{(ii)} & \text{if } 1\le k\le 2n-1, \\ & \text{then } w=(w_{10}w_{11})\cdots(w_{n,0}w_{n,1}), \\ & \text{where } w_{i0}\in\{w_0,w_0^*\}, w_{i1}\in\{w_1,w_1^*\}, \text{ and} \\ & \operatorname{des}(w)=|\{w_1,w_1^*\}|=2, \\ \text{(iii)} & \text{if } k=2n, \text{ then } w=(w_0^*w_1^*)^n, \text{ and} \\ & \operatorname{des}(w)=|\{w_1^*\}|=1. \end{cases}$$

Further, for

$$w=R^{-2n-m}T^{-n}=(w_1w_1^*)^m(w_0^*w_1^*)^n \quad (m\ge1),$$

$\operatorname{des}(w)=|\{w_1^*,w_0\}|=2$, and for $w=R^mT^{-n}=(w_1^*w_1)^m(w_0w_1)^n$ $(m\ge1)$, $\operatorname{des}(w)=|\{w_1,w_0^*\}|=2$. From the above, we have;

$$W(t,q)_{(II)}=\sum_{n\ge1}2tq^{2n}+\sum_{n\ge1}(2n-1)t^2q^{2n}+\sum_{n\ge1,m\ge1}2t^2q^{2n+2m}.$$

(III) $T^nw_1=(w_1w_0)^nw_1$ $(n\ge0)$.

For $k\ge0$, we consider $w=R^kT^nw_1$.

$$\begin{cases} \text{(i)} & \text{if } k=0, \text{ then } w=(w_1w_0)^nw_1, \text{ and} \\ & \operatorname{des}(w)=|\{w_1\}|=1, \\ \text{(ii)} & \text{if } 1\le k\le 2n, \\ & \text{then } w=(w_{11}w_{10})\cdots(w_{n1}w_{n0})w_{n+1,1}, \\ & \text{where } w_{i1}\in\{w_1,w_1^*\}, w_{i0}\in\{w_0,w_0^*\}, \text{ and} \\ & \operatorname{des}(w)=|\{w_1,w_1^*\}|=2, \\ \text{(iii)} & \text{if } k=2n+1, \text{ then } w=(w_1^*w_0^*)^nw_1^*, \text{ and} \\ & \operatorname{des}(w)=|\{w_1^*\}|=1. \end{cases}$$

Further, for $w=R^{2n+1+m}T^nw_1=(w_0w_0^*)^m(w_1^*w_0^*)^nw_1^*=w_1^*(w_1w_1^*)^m(w_0^*w_1^*)^n=w_1^*(w_0^*w_1^*)^n(w_0^*w_0)^m$ $(m\ge1)$, $\operatorname{des}(w)=|\{w_1^*,w_0\}|=2$, and for $w=R^{-m}T^nw_1=(w_0^*w_0)^m(w_1w_0)^nw_1=w_1(w_1^*w_1)^m(w_0w_1)^n=w_1(w_0w_1)^n(w_0w_0^*)^m$ $(m\ge1)$, $\operatorname{des}(w)=|\{w_1,w_0^*\}|=2$. From the above, we have;

$$W(t,q)_{(\mathrm{III})}=\sum_{n\ge0}2tq^{2n+1}+\sum_{n\ge0}2nt^2q^{2n+1}+\sum_{n\ge0,m\ge1}2t^2q^{2n+2m+1}.$$

(IV) $T^{-n}w_1=(w_0w_1)^{n-1}w_0$ $(n\ge1)$.

For $k\ge0$, we consider $w=R^{-k}T^{-n}w_1$.

$$\begin{cases} \text{(i)} & \text{if } k=0, \text{ then } w=(w_0w_1)^{n-1}w_0, \text{ and} \\ & \operatorname{des}(w)=|\{w_0\}|=1, \\ \text{(ii)} & \text{if } 1\le k\le 2n-2, \text{ then} \\ & w=(w_{10}w_{11})\cdots(w_{n-1,0}w_{n-1,1})w_{n,0}, \\ & \text{where } w_{i0}\in\{w_0,w_0^*\}, w_{i1}\in\{w_1,w_1^*\}, \text{ and} \\ & \operatorname{des}(w)=|\{w_0,w_0^*\}|=2, \\ \text{(iii)} & \text{if } k=2n-1, \text{ then } w=(w_0^*w_1^*)^{n-1}w_0^*, \text{ and} \\ & \operatorname{des}(w)=|\{w_0^*\}|=1. \end{cases}$$

Further, for $w=R^{-(2n-1)-m}T^{-n}w_1=(w_1w_1^*)^m(w_0^*w_1^*)^{n-1}w_0^*=(w_0^*w_1^*)^{n-1}(w_0^*w_0)^mw_0^*=(w_0^*w_1^*)^{n-1}w_0^*(w_1^*w_1)^m$ $(m\ge1)$, $\operatorname{des}(w)=|\{w_0^*,w_1\}|=2$, and for $w=R^mT^{-n}w_1=(w_1^*w_1)^m(w_0w_1)^{n-1}w_0=(w_0w_1)^{n-1}(w_0w_0^*)^mw_0=$

$(w_0w_1)^{n-1}w_0(w_1w_1^*)^m$ $(m \geq 1)$, $\mathrm{des}(w) = |\{w_0, w_1^*\}| = 2$. From the above, we have;

$$W(t,q)_{(\mathrm{IV})} = \sum_{n\geq 1} 2tq^{2n-1} + \sum_{n\geq 1}(2n-2)t^2q^{2n-1} + \sum_{n\geq 1, m\geq 1} 2t^2q^{2n+2m-1}.$$

Proposition 2.2. *The q-Eulerian distribution of $W\left(A_1^{(1,1)}\right)$ with the above generator system is given as follows*:

$$W(t,q) = \sum_{w\in W\left(A_1^{(1,1)}\right)} t^{\mathrm{des}(w)}q^{l(w)} = (1-q+2qt)^2/(1-q)^2.$$

Proof. From the above (I)–(IV),

$$\begin{aligned} W(t,q) = 1 &+ \sum_{m\geq 1} 2t^2q^{2m} + \sum_{n\geq 1} 2tq^{2n} \\ &+ \sum_{n\geq 1}(2n-1)t^2q^{2n} + \sum_{n\geq 1, m\geq 1} 2t^2q^{2n+2m} \\ &+ \sum_{n\geq 1} 2tq^{2n} + \sum_{n\geq 1}(2n-1)t^2q^{2n} \\ &+ \sum_{n\geq 1, m\geq 1} 2t^2q^{2n+2m} + \sum_{n\geq 0} 2tq^{2n+1} \\ &+ \sum_{n\geq 0} 2nt^2q^{2n+1} + \sum_{n\geq 0, m\geq 1} 2t^2q^{2n+2m+1} \\ &+ \sum_{n\geq 1} 2tq^{2n-1} + \sum_{n\geq 1}(2n-2)t^2q^{2n-1} \\ &+ \sum_{n\geq 1, m\geq 1} 2t^2q^{2n+2m-1} \\ = \frac{(1-q+2qt)^2}{(1-q)^2}. \end{aligned}$$

□

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