Some results on Bernstein-Sato polynomials for parametric analytic functions

By Rouchdi Bahloul

Laboratoire de Mathématiques, Université de Versailles St-Quentin-en-Yvelines,
45 avenue des Etats-Unis-Bâtiment Fermat, 78035 Versailles, France

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Abstract: This is the second part of a work dedicated to the study of Bernstein-Sato polynomials for several analytic functions depending on parameters. In this part, we give constructive results generalizing previous ones obtained by the author in the case of one function. We also make an extensive study of an example for which we give an expression of a generic (and under some conditions, a relative) Bernstein-Sato polynomial.

Key words: Bernstein-Sato polynomial; deformation of singularities; generic standard bases.

Let $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ be compact polydiscs centered at the origin, $Z = X \times Y$ and $f = (f_1, \dots, f_p)$ ($p \geq 2$) an analytic map from X to $\mathbf{C}^p$. We are interested in the study of Bernstein-Sato polynomials of $f(x, y_0)$ when y_0 moves through Y. Our work is related to the notion of generic Bernstein-Sato polynomials as in Briançon *et al.* [10] (for $p = 1$) and Biosca [7]. Herein we shall adopt a more constructive method as in Bahloul [4] (where the case $p = 1$ was treated), based on the first part [5] and Bahloul [2].

Our goal is to give analogous results to [4]. However, since the construction in [2] is entirely algorithmic only when $p = 2$, a part of the results herein shall be shown only for $p = 2$. It would be a nice result if one could wholly achieve [2] in an algorithmic way (here "algorithmic" means "in an infinite way"). Note that a similar question was treated in the case of polynomials f_j in Bahloul [1] with direct methods while constructive methods were used in Leykin [16] (for $p = 1$) and Briançon, Maisonobe [13] (for $p \geq 1$).

Note. If $\mathcal{O}_{\mathbf{C}^{n+m}}$ denotes the sheaf of analytic functions on $\mathbf{C}^{n+m}$, we shall identify $\mathcal{O}_Z$ with the germ $\mathcal{O}_{\mathbf{C}^{n+m},0}$. Sometimes, we will reduce Z without an explicit mention so that $\mathcal{O}_Z$ shall be identified with the set $\mathcal{O}_{\mathbf{C}^{n+m}}(U)$ of sections of $\mathcal{O}_{\mathbf{C}^{n+m}}$ on an open (poly)disc $0 \in U \subset Z$.

1. Main results. $\mathcal{D}_{Z/Y}$ denotes the ring of relative differential operators. It is the subring of $\mathcal{D}_Z$ made of elements without derivations ∂_{y_i}. Let us write $s = (s_1, \dots, s_p)$ and $\partial_t = (\partial_{t_1}, \dots, \partial_{t_p})$. Following [13], define $\mathbf{C}\langle s, \partial_t\rangle$ as the algebra $\mathbf{C}[s, \partial_t]$ with the relations $\partial_{t_j} s_j = s_j \partial_{t_j} - \partial_{t_j}$ ($j = 1, \dots, p$) and set $\mathcal{D}_Z\langle s, \partial_t\rangle = \mathcal{D}_Z \otimes \mathbf{C}\langle s, \partial_t\rangle$. If $t = (t_1, \dots, t_p)$ are new indeterminates, the identification $s_j = -\partial_{t_j} t_j$ gives the inclusions of rings: $\mathcal{D}_Z[s] \subset \mathcal{D}_Z\langle s, \partial_t\rangle \subset \mathcal{D}_{Z\times\mathbf{C}^p}$. This identification comes from the fact that the free $\mathcal{O}_Z[1/F, s]$-module $\mathcal{O}_Z[1/F, s] \cdot f^s$ (here $F = f_1 \dots f_p$ and $f^s = f_1^{s_1} \cdots f_p^{s_p}$) is a $\mathcal{D}_{Z\times\mathbf{C}^p}$-module and the action of s_j coincides with that of $-\partial_{t_j} t_j$ (see Malgrange [17]).

For a given set of (germs at 0 of) analytic functions $g = (g_1, \dots, g_p)$ on X, $\mathcal{B}(g)$ shall denote the ideal of Bernstein-Sato of g (at $x = 0$): it is the set of $b(s) \in \mathbf{C}[s]$ satisfying $b(s) g^s \in \mathcal{D}_X[s] g^{s+1}$ (here $g^{s+1} := \prod g_j^{s_j+1}$). This ideal is not zero (Sabbah [19]) and in fact it contains a polynomial of the form $\prod(l_1 s_1 + \cdots + l_p s_p + a)$ with $l_j \in \mathbf{N}$ and $a \in \mathbf{Q}_{>0}$ (Gyoja [15]).

Remark 1.1. If for any $j = 1, \dots, p$, $g_j^{-1}(0) \not\subseteq \bigcup_{k\neq j} g_k^{-1}(0)$ then $\mathcal{B}(g) \subset \mathbf{C}[s] \cdot \prod_j (s_j + 1)$. Indeed, it suffices to specialize $s_j = -1$ in a functional equation.

When the g_j are in $\mathbf{k}[[x]]$ for some field $\mathbf{k}$ (of characteristic 0) we can also consider $\mathcal{B}(g) \subset \mathbf{k}[s]$ the ideal defined by the same relation where we replace $\mathcal{D}_X$ by $\hat{\mathcal{D}}_x(\mathbf{k}) = \mathbf{k}[[x]][\partial_x]$. It is well known [11] that given $g \in (\mathcal{O}_X)^p \subset \mathbf{C}[[x]]$, the formal Bernstein-Sato ideal coincides with the analytic one.

For $g \in (\mathbf{k}[[x]])^p$ it is still an open question

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whether or not $\mathcal{B}(g)$ is zero. We know it is not zero only when $p = 1$ (Björk [9]).

Let us come back to our situation. We retain the notations of part 1 [5]. Set $\mathcal{C} = \mathcal{O}_Y$ and $\mathcal{Q} \in \mathrm{Spec}(\mathcal{C})$. Each f_j is viewed as an element of $\mathcal{C}[[x]]$ and we consider the specialization $(f_j)_\mathcal{Q} \in \mathrm{Frac}(\mathcal{C}/\mathcal{Q})[[x]]$ to $\mathcal{Q}$ so that $\mathcal{B}((f)_\mathcal{Q})$ is an ideal of $\mathrm{Frac}(\mathcal{C}/\mathcal{Q})[s]$.

Theorem 1. *For any $b(s) \in \mathbf{C}[s]$, the following conditions are equivalent*:

(i) $b(s) \in \mathcal{B}((f)_\mathcal{Q})$.

(ii) $\exists h(x,y) \in \mathcal{O}_Z$ *with* $h(0,y) \notin \mathcal{Q}$ *such that*

$$h(x,y)b(s)f^s \in \mathcal{D}_{Z/Y}[s]f^{s+1} + \mathcal{Q}\mathcal{D}_{Z/Y}\langle s, \partial_t\rangle f^s.$$

(iii) $\exists c(y) \in \mathcal{O}_Y \smallsetminus \mathcal{Q}$ *such that for any* $y_0 \in V(\mathcal{Q}) \smallsetminus V(c)$, $b(s) \in \mathcal{B}(f(x,y_0))$.

Proof. The proof is similar to that of [4, Th. 1]. Let us give its main lines. For (ii) $\Rightarrow$ (i), it suffices to specialize to $\mathcal{Q}$, while (ii) $\Rightarrow$ (iii) is trivial by taking $c(y) = h(0,y)$.

Let us introduce two ideals: I_0 the ideal of $\hat{\mathcal{D}}_x(\mathcal{C})\langle s, \partial_t\rangle = \mathcal{C}[[x]][\partial_x] \otimes \mathcal{C}\langle s, \partial_t\rangle$ generated by the $s_j + f_j\partial_{t_j}$, $j = 1, \dots, p$, and the $\partial_{x_i} + \sum_{j=1}^p (\partial f_j/\partial x_i)\partial_{t_j}$, $i = 1, \dots, n$; and $I'_0 \subset \hat{\mathcal{D}}_x(\mathcal{C})[s]$ defined as

$$I'_0 = (I_0 + \hat{\mathcal{D}}_x(\mathcal{Q})\langle s, \partial_t\rangle) \cap \mathcal{D}_X(\mathcal{C})[s] + \mathcal{D}_X(\mathcal{C})[s] \cdot F.$$

For any $y_0 \in Y$, $I_{0|y=y_0}$ is the annihilator in $\hat{\mathcal{D}}_x(\mathbf{C})\langle s, \partial_t\rangle$ of $\prod_j f_j(x,y_0)^{s_j}$ (see e.g. [3, sect. 4] for $p = 1$, the proof for $p \geq 2$ is the same).

Moreover, we have the following (by using an arbitrary generic standard basis of I'_0): $(I'_0)_\mathcal{Q}$ equals

$$(I_0)_\mathcal{Q} \cap \hat{\mathcal{D}}_x(\mathrm{Frac}(\mathcal{C}/\mathcal{Q}))[s] + \hat{\mathcal{D}}_x(\mathrm{Frac}(\mathcal{C}/\mathcal{Q}))[s] \cdot (F)_\mathcal{Q}.$$

As a consequence, for $b \in \mathbf{C}[s]$, $b \in \mathcal{B}((f)_\mathcal{Q})$ if and only if $b \in (I'_0)_\mathcal{Q}$.

Now assume we have (iii). Consider the division modulo $\mathcal{Q}$ of b by (a generic standard basis of) I'_0 (see [5, Prop. 2.2] and [4, Prop. 3.5]) and denote by R the remainder mod$\mathcal{Q}$. It follows that for a generic $y_0 \in V(\mathcal{Q})$, $R_{|y=y_0}$ is in $I'_{0|y=y_0}$. This is possible only if R is zero modulo $\mathcal{Q}$, thus

$$b \in \hat{\mathcal{D}}_x(\mathcal{C}[c^{-1}])[s] \cdot I'_0 + \hat{\mathcal{D}}_x(\mathcal{C}[c^{-1}])[s] \cdot F$$

for some $c \in \mathcal{C} \smallsetminus \mathcal{Q}$. Specializing this relation to $\mathcal{Q}$, we get $b = (b)_\mathcal{Q} \in (I'_0)_\mathcal{Q}$. Thus (i) is satisfied.

Now assume we have (i), which means that $b \in (I'_0)_\mathcal{Q}$. Let us consider the division modulo $\mathcal{Q}$ of b by (an arbitrary generic standard basis of) I'_0. The remainder is zero modulo $\mathcal{Q}$, which means that

$$b \in \hat{\mathcal{D}}_x(\mathcal{C}[c^{-1}])[s] \cdot I'_0 + \hat{\mathcal{D}}_x(\mathcal{Q}[c^{-1}])[s]$$

for some $c \in \mathcal{O}_Y \smallsetminus \mathcal{Q}$. Applying b to f^s, we obtain a formal functional equation of the form

$$bf^s \in \hat{\mathcal{D}}_x(\mathcal{O}_Y[c^{-1}])[s]f^{s+1} + \hat{\mathcal{D}}_x(\mathcal{Q}[c^{-1}])\langle s, \partial_t\rangle f^s.$$

We may then pass from the formal to the analytic setting (following the same arguments as in the last section of [4]) and we get (ii). □

We still don't know whether or not $\mathcal{B}((f)_\mathcal{Q}) \cap \mathbf{C}[s]$ is zero.

Theorem 2. *Here* $p = 2$ (*see the comments in* §2 *of the introduction*). *There exists a non zero polynomial* $b(s)$ *of the form* $\prod(l_1s_1 + \cdots + l_ps_p + a)$ *with* $l_j \in \mathbf{N}$ *and* $a \in \mathbf{Q}_{>0}$, *that belongs to* $\mathcal{B}((f)_\mathcal{Q})$.

The proof will be given in Section 4.

As a consequence: there exists a finite stratification $Y = \bigcup W$ into locally closed subsets W and polynomials $b_W(s)$ of the above form such that for any $y_0 \in W$, $b_W(s) \in \mathcal{B}(f(x,y_0))$.

Consider the lcm of the b_W and denote it by b_{comp} then for any $y_0 \in Y$, b_{comp} is a Bernstein-Sato polynomial of $f(x,y_0)$. Here b_{comp} should be read "comprehensive Bernstein-Sato polynomial." It is clear that any "relative Bernstein-Sato polynomial" is comprehensive but the converse is obviously wrong since a relative Bernstein-Sato polynomial does not exist in general, even when $p = 1$ (see e.g. [8] for the definition of a relative Bernstein-Sato polynomial, see also [12] for general results on the subject in the hypersurface case).

Corollary 3. *Here* p *is not necessarily* 2. *Take* $n = 2$ *and suppose that for a generic* y_0 *in* $V(\mathcal{Q})$, $f_1(x,y_0), \dots, f_p(x,y_0)$ *are irreducible and pairwise relatively prime. Take* $b(s) \in \mathbf{C}[s]$ *then* $b(s) \in \mathcal{B}((f)_\mathcal{Q})$ *if and only if there exists* $H(y) \in \mathcal{O}_Y \smallsetminus \mathcal{Q}$ *with*

$$H(y)b(s) \in \mathcal{D}_{Z/Y}[s]f^{s+1} + \mathcal{Q}\mathcal{D}_{Z/Y}\langle s, \partial_t\rangle f^s.$$

This means that $b(s)$ is a "generic Bernstein-Sato" polynomial in the sense of Biosca [7] (notice that in previous works on this subject, the notion of generic Bernstein-Sato polynomial is defined only when $\mathcal{Q} = (0)$, see e.g. loc. cit. and its references).

The assumptions of this corrolary mean that the relative singular locus $V(\partial F/\partial x_1, \dots, \partial F/\partial x_n, F)$ projects to 0 by the projection $X \times Y \to X$ when we restrict ourself to $X \times U$ and U is a Zariski open set of $V(\mathcal{Q})$.

Let us give a:

Sketch of proof of Cor. 3. The "if" sense is trivial. Let us prove the converse. We don't give all the details of the proof for it is analogous to that of [4, Cor. 2]. Denote by J the ideal of $\mathcal{O}_Z$ generated by F and the $\partial F/\partial x_i$'s. The hypothesis can be rephrased as follows:

$$V\big(\sqrt{J+\mathcal{O}_Z\cdot\mathcal{Q}}:h_0\big)\subset(0)\times V(\mathcal{Q})$$

in $Z=X\times Y$, for some $h_0(y)\in\mathcal{O}_Y\smallsetminus\mathcal{Q}$. Thus the zero locus of $\sqrt{J+\mathcal{O}_Z\cdot\mathcal{Q}}:h_0+\mathcal{O}_Z\cdot h$ is included in the zero set of $h(0,y)$, where h is obtained from Th. 1 (ii). As a consequence $H:=(h_0h(0,y))^k$ is in $J+\mathcal{O}_Z\cdot\mathcal{Q}+\mathcal{O}_Z\cdot h$, for some $k\in\mathbf{N}$, by using Hilbert's Nullstellensatz. This H is in $\mathcal{O}_Y\smallsetminus\mathcal{Q}$.

Now, for a generic y_0 in $V(\mathcal{Q})$, $b(s)$ is a Bernstein-Sato polynomial for $f(x,y_0)$, thus by assumption and Rem. 1.1, $\prod_1^p(s_j+1)$ divides $b(s)$. Let us write $b(s)=\prod_1^p(s_j+1)\cdot\tilde{b}(s)$. For $i=1,\dots,n$, we have:

$$\begin{aligned}
&b(s)\frac{\partial F}{\partial x_i}f^s\\
&=\tilde{b}(s)\prod_{k=1}^p(s_k+1)\left(\sum_{j=1}^p\frac{\partial f_j}{\partial x_i}\frac{F}{f_j}\right)f^s\\
&=\tilde{b}(s)\sum_{j=1}^p\left(\prod_{k\neq j}(s_k+1)\right)(s_j+1)\frac{\partial f_j}{\partial x_i}\frac{1}{f_j}f^{s+1}\\
&=\tilde{b}(s)\sum_{j=1}^p\left(\prod_{k\neq j}(s_k+1)\right)\partial_{x_i}\cdot f^{s+1}.
\end{aligned}$$

From this equality, and relation (ii) in Th. 1, we get the desired equation with $H(y)$. □

2. An example related to [3]. Let us consider the following example:

$$\begin{aligned}
f_1(x,y)&=c_1(y)x_1^a+c_2(y)x_2^b+g_1(x_1,x_2,y)\\
f_2(x,y)&=c_3(y)x_1^c+c_4(y)x_2^d+g_2(x_1,x_2,y)
\end{aligned}$$

with $y=(y_1,\dots,y_m)$, $x=(x_1,x_2)$, $a,b,c,d\in\mathbf{N}_{>0}$. Here, $c_i\in\mathcal{O}_Y$ and $g_i(x,y)\in\mathcal{O}_Z$. We assume that $C(y):=\prod_i c_i(y)$ is not zero, and we work with $\mathcal{Q}=(0)$ so that $V(\mathcal{Q})=Y$.

Consider the weight vectors $\alpha_1=(b,a)$, $\alpha_2=(d,c)$ on the variables (x_1,x_2), with $bc>ad$. The weight of an element g in $\mathcal{O}_Z$ for α_i, denoted by $\rho_{\alpha_i}(g)$, is the minimum of the α_i-degrees in x of the monomials of g.

We assume that $\rho_{\alpha_i}(g_i)>\rho_{\alpha_i}(f_i)$, $i=1,2$. As a consequence, for any y_0 with $C(y_0)\neq 0$, it is easy to check that $f_1(x,y_0)$ and $f_2(x,y_0)$ are irreducible and relatively prime, so f satisfies the assumptions of Cor. 3.

On the other hand, for any y_0 with $C(y_0)\neq 0$, the main result of Bahloul [3] applies. Put $N_1=2ab+ad-2a-2b$, $N_2=2cd+ad-2c-2d$, $W_1=\{\deg_{\alpha_1}(z)\}$ (resp. $W_2=\{\deg_{\alpha_2}(z)\}$), z running over the monomials with $\deg_{\alpha_1}(z)\leq N_1+\rho_{\alpha_1}(f_2)$ (resp. $\deg_{\alpha_2}(z)\leq N_2+\rho_{\alpha_2}(f_1)$), and $b(s_1,s_2)=(s_1+1)(s_2+1)\prod_{\rho_1\in W_1}(abs_1+ads_2+a+b+\rho_1)\prod_{\rho_2\in W_2}(ads_1+cds_2+c+d+\rho_2)$.

By [3, Prop. 1], for any $y_0\in Y$ with $C(y_0)\neq 0$, the polynomial $b(s_1,s_2)$ is in $\mathcal{B}((f_1,f_2)(x,y_0))$, that is, b satisfies Th. 1 (iii). Applying Cor. 3, we get:

$$H(y)b(s_1,s_2)f_1^{s_1}f_2^{s_2}\in\mathcal{D}_{Z/Y}[s_1,s_2]f_1^{s_1+1}f_2^{s_2+1}$$

for some non zero $H(y)\in\mathcal{O}_Y$. This means that b is a generic Bernstein-Sato polynomial in the usual sense. If we look at the details of the proof of (iii) ⇒ (ii) in Th. 1 and the proof of Cor. 3, we notice that the $H(y)$ obtained in this corollary is of the form $C(y)^k$ for some $k\in\mathbf{N}$, thus:

Proposition 2.1. *For some $k\in\mathbf{N}$, we have*

$$C(y)^kb(s_1,s_2)f_1^{s_1}f_2^{s_2}\in\mathcal{D}_{Z/Y}[s_1,s_2]f_1^{s_1+1}f_2^{s_2+1}.$$

As a consequence, if $C(y)$ is invertible (i.e. $C(0)\neq 0$) then b is a relative Bernstein-Sato polynomial. In fact, we have a more precise statement:

Proposition 2.2. *Set $C'(y)=c_1(y)c_4(y)$. If C' is invertible then the polynomial b above is a relative Bernstein-Sato polynomial*:

$$b(s_1,s_2)f_1^{s_1}f_2^{s_2}\in\mathcal{D}_{Z/Y}[s_1,s_2]f_1^{s_1+1}f_2^{s_2+1}.$$

It is a direct consequence of the following result.

Claim 2.3. *The polynomial b above satisfies*:

$$(1+p){C'}^kb(s_1,s_2)f_1^{s_1}f_2^{s_2}\in\mathcal{D}_{Z/Y}[s_1,s_2]f_1^{s_1+1}f_2^{s_2+1}$$

for some $p\in\sum_{i=1}^n\mathcal{O}_Z[C'^{-1}]\cdot x_i$ and some $k\in\mathbf{N}$.

Proof of the claim. The proof follows [3]. Let us first review it and then explain how it can be adapted to our situation. In [3], the data are analytic functions f_1,f_2 satisfying some conditions. For example, $f_1=x_1^a+x_2^b+g_1(x)$ and $f_2=x_1^c+x_2^d+g_2(x)$ with $\rho_{\alpha_i}(g_i)>\rho_{\alpha_i}(f_i)$. We define $\xi_{i_1,i_2}=\prod_{k=0}^{i_1-1}(s_1-k)\prod_{k=0}^{i_2-1}(s_2-k)f_1^{s_1-i_1}f_2^{s_2-i_2}$ for $(i_1,i_2)\in\mathbf{N}^2$, $\xi_{-1,0}=f_1^{s_1+1}f_2^{s_2}$, $\xi_{0,-1}=f_1^{s_1}f_2^{s_2+1}$, $\xi_{-1,-1}=f_1^{s_1+1}f_2^{s_2+1}$. Then we attach α_i-weights to the elements of $\mathcal{D}_X[s_1,s_2]\xi_{i_1,i_2}$ (see [3, Def. 1.3]), by defining $\rho_{\alpha_i}\big(\sum_{\beta,k,l}\partial_x^\beta s_1^k s_2^l u_{\beta kl}(x)\xi_{i_1,i_2}\big)$ as the minimum of $\rho_{\alpha_i}(u_{\beta kl}(x)) - i_1\rho_{\alpha_i}(f_1) - i_2\rho_{\alpha_i}(f_2)$.

On the other hand, we introduce the ideals of $\mathcal{O}_X$: $I = \langle f_1, f_2\rangle$, $I_1 = \langle f_1, J\rangle$ and $I_2 = \langle f_2, J\rangle$. Here J is the determinant of the jacobian matrix of $(f_1, f_2)(x_1, x_2)$. We show that these ideals have a finite colength lower than N_1 and N_2. For this purpose we use divisions and standard bases settings. The local order used in the divisions is such that the leading terms of f_1, f_2 and J are x_1^a, x_2^d, $adx_1^{a-1}x_2^{d-1}$ respectively.

Step 1. The first step of the proof is to show that applying b to $\xi_{0,0} = f_1^{s_1}f_2^{s_2}$ gives rise to a (finite) sum of elements $(s_1+1)(s_2+1)P(s)\xi_{i_1,i_2}$ with α_i-weight $> N_i$.

Step 2. By division first by I and then by the I_i's we can go down from $\mathcal{D}_X[s]\xi_{i_1,i_2}$ to $\mathcal{D}_X[s]\xi_{i_i-1,i_2}$ and $\mathcal{D}_X[s]\xi_{i_1,i_2-1}$ while the α_i-weight is conserved. This enables an induction on i_1 and i_2, so that we can go back to $\xi_{0,0} = f_1^{s_1}f_2^{s_2}$.

Now let us see how the proof of [3] can be adapted to prove our claim.

We work in a formal setting so that f_i are viewed in $\mathcal{O}_Y[[x]]$. Step 1 can be done without any problems. In Step 2, we shall do divisions by I (resp. I_1, I_2). But the leading terms of f_1, f_2, J are $c_1x_1^a$, $c_4x_2^d$, $c_1c_4adx_1^{a-1}x_2^{d-1}$ (see Propr. 2.5 and the proof of Aff. 4.1 in [3]) so all the divisions will take place in $\mathcal{O}_Y[{C'}^{-1}][[x]]$. Therefore, the equation obtained will be of the form:

$$b(s_1,s_2)f_1^{s_1}f_2^{s_2} \in \hat{\mathcal{D}}(\mathcal{O}_Y[{C'}^{-1}])[s_1,s_2]f_1^{s_1+1}f_2^{s_2+1}.$$

As above, we may pass from the formal to the analytic setting to conclude. □

3. Recalls and preliminaries on Bernstein-Sato polynomials. In order to prove Th. 2, we shall review some results.

For more details, see [2]. The system of coordinates (x,t) being fixed, we denote by V_j the V-filtration associated with the hypersurface $t_j = 0$, $j = 1,\dots,p$, on $\mathcal{D}_{X\times\mathbf{C}^p}$ (we can see it as the natural filtration associated with the weight vector also denoted V_j where the weight of t_j and ∂_{t_j} are -1 and 1 respectively, and the weight is zero for the other symbols). For $L = (l_1,\dots,l_p)$ in $(\mathbf{R}_{\geq 0})^p$, we denote by V^L the filtration $\sum_j l_jV_j$ and gr^L the associated graded ring.

Given $g = (g_1,\dots,g_p)$ analytic on X, we define I the annihilator of g^s in $\mathcal{D}_{X\times\mathbf{C}^p}$. The ideal $\mathcal{B}_L$ is then defined as the set of $c(s) \in \mathbf{C}[s_1,\dots,s_p]$ with the relation

$$c(s)g^s \in V^L_{<0}(\mathcal{D}_{X\times\mathbf{C}^p})g^s.$$

Then $b_L = b_{L,g}$ (if it is not zero) is the monic polynomial $e(\lambda)$ in one variable of the least degree satisfying: $e(L(s)) \in \mathcal{B}_L$. Here $L(s) = \sum_j l_js_j$. This polynomial is not zero (Sabbah [19] and has roots in $\mathbf{Q}_{<0}$ Gyoja [17]). It can be seen as the Bernstein-Sato polynomial of g in the "direction" L. Notice that in the algebraic case, $b_{(1,\dots,1)}$ coincides with the b-function considered in Budur *et al.* [16].

Now we can consider the restriction $\mathcal{E}_V(h(I))$ to the space $\sum_j \mathbf{R}_{\geq 0}V_j$ of the analytic Gröbner fan of I, for which we denote by $Sq(\mathcal{E}_V(h(I)))$ the 1-skeleton. This restriction leaves in $(\mathbf{R}_{\geq 0})^p$ and this skeleton is in $\mathbf{N}^p$ (because the Gröbner fan is rational).

Theorem 3.1 ([19] and [2]). *There exists $\kappa \in \mathbf{N}^p$ such that the polynomial*

$$b(s) = \prod_{L\in Sq(\mathcal{E}_V(h(I)))}\ \prod_{-L(\kappa+(1,\cdots,1))<k\leq 0} b_{L,g}(L(s)-k)$$

is Bernstein-Sato polynomial of g.

Remark 3.2.

- In Sabbah [19], the author shows that there exists $\kappa \in \mathbf{N}^p$ satisfying a certain property, say $(\mathcal{P})$. Then his shows that if κ satisfies $(\mathcal{P})$ then it satisfies Th. 3.1.
- When $p = 2$, in [2], we construct explicitely some κ making $(\mathcal{P})$ true. The construction goes as follows: Let $L_1,\dots,L_q \in \mathbf{N}^p$ be such that $C_{L_i}(h(I))$ are the maximal cones of the (open) fan $\mathcal{E}_V(h(I))$. Let G_i be the reduced standard basis of $h(I)$ for an order $\prec^h_{L_i}$ adapted to L_i and define κ^1 as the maximum of the $\mathrm{ord}^{V_1}(P) - \mathrm{ord}^{V_1}(\mathrm{lm}_{\prec^h_{L_i}}(P))$ where P runs over all the element of all the G_i's. Then $\kappa = (\kappa^1, 0)$ satisfies property $(\mathcal{P})$. Here ord^{V_1} means the order with respect to the filtration V_1 (see [2]).
  Notice that this κ depends (only) on two monomials of each element of the standard bases. Thus it depends on a finite number of monomials.

Lemma 3.3.

(a) *With the identification $s_j = -\partial_{t_j}t_j$, we have* $\mathcal{B}_L(g) = \mathrm{gr}^L(I) \cap \mathbf{C}[s_1,\dots,s_p]$.

(b) $b_{L,g}(L(s))$ *is the monic generator of* $\mathrm{gr}^L(I) \cap \mathbf{C}[l_1s_1+\cdots+l_ps_p]$.

The lemma is a straightforward consequence of the definitions.

4. Proof of Theorem 2. The proof shall be partially sketched because it uses the same method as in [4].

4.1. Formal algorithm for b_L. Given $L \in (\mathbf{R}_{\geq 0})^p$, we give an algorithm for computing the polynomial b_L for formal power series $g_1, \dots, g_p \in \mathbf{k}[[x]]$, $\mathbf{k}$ denotes a field of characteristic 0.

Lemma 4.1. *Let I be an ideal in $\hat{\mathcal{D}}_{x,t}$. Let $2 \leq k \leq p$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, p\}$ such that $l_j = 0$ iff $j = j_i$ with $1 \leq i \leq k$ (if none of the l_j is 0 then put $k = 1$). Then $\mathrm{gr}^L(\hat{\mathcal{D}}_{x,t})$ is canonically isomorphic to $\mathbf{k}[[x, t_{j_1}, \dots, t_{j_{k-1}}]]\langle t_{j_k}, \dots, t_{j_p}, \partial_t, \partial_x\rangle$, where the commutation relations are obtained from $\hat{\mathcal{D}}_{x,t}$ by restriction.*

The proof is straightforward. Thus this graded ring is a subring of $\hat{\mathcal{D}}_{x,t}$ and it can be constructed as in [4, Section 3]. Therefore all the results of loc. cit. about (generic) standard bases apply.

In the following, in order to simplify, we assume $\{j_1, \dots, j_{k-1}\} = \{1, \dots, k-1\}$, i.e. $L = (0, l_k, \dots, l_p)$. Consider the following ideals:

(0) $I = \sum_{j=1}^p \hat{\mathcal{D}}_{x,t}(t_j - g_j) + \sum_{i=1}^n \hat{\mathcal{D}}_{x,t}(\partial_{x_i} + \sum_{j=1}^p (\partial g_j/\partial x_i)\partial_{t_j})$.
This ideal is the annihilator of g^s in $\hat{\mathcal{D}}_{x,t}$.

(1) $I_1 = \mathrm{gr}^L(I)$ in $\mathbf{k}[[x, t_{j_1}, \dots, t_{j_{k-1}}]]\langle t_{j_k}, \dots, t_{j_p}, \partial_t, \partial_x\rangle$.

(2) $I_2 = I_1 \cap \mathbf{k}[[x, t_1, \dots, t_{k-1}]]\langle t_k, \dots, t_p, \partial_{t_k}, \dots, \partial_{t_p}\rangle$.
It is an elimination of the "global" variables ∂_{x_i} and ∂_{t_j} for $j = 1, \dots, k-1$.

(3) We introduce a new indeterminate λ and we consider the ring

$$\mathbf{k}[[x, t_1, \dots, t_{k-1}]]\langle t_k, \dots, t_p, \partial_{t_k}, \dots, \partial_{t_p}\rangle[\lambda]$$

where the new relations are: $[\lambda, x_i] = [\lambda, t_j] = 0$ for any i and any $1 \leq j \leq k-1$ and for $j \geq k$ $[t_j, \lambda] = l_j t_j$ and $[\partial_{t_j}, \lambda] = -l_j \partial_{t_j}$. In other terms, λ behaves like $l_k s_k + \cdots + l_p s_p$ where $s_j = -\partial_{t_j} t_j$.
We consider then the previous ideal in this ring and we put

$$I_3 = I_2 \cap \mathbf{k}[[x, t_1, \dots, t_{k-1}]][\lambda].$$

We have eliminated the "global" variables t_j and ∂_{t_j} for $j \geq k$. Notice that now we have a commutative setting.

(4) $I_4 = I_3 \cap \mathbf{k}[\lambda]$. We eliminate the "local" variables x_i and t_j.

Lemma 3.3 asserts that the monic generator of I_4 (if it is not zero) is the polynomial $b_{L,g}$.

The reason why we need to go through step (3) is that we know how to go from a given ideal $\mathcal{I} \subset \mathbf{k}[[y_1, \dots, y_m]][\lambda_1, \dots, \lambda_q]$ to the ideal $\mathcal{I} \cap \mathbf{k}[\lambda_1, \dots, \lambda_q]$ only when $q = 1$. We can find such an algorithm in [4, 4.1] (which is a variant of Oaku's [18, Algo. 4.5]).

Details for step (1). All the steps but step (1) consist in the elimination of global or local variables. The elimination of global variables can be done as in [4, Prop. 3.8] whereas the local elimination is described in [4, 4.1]. Let us discuss step (1). As in [5], we consider $h(I) \subset \hat{\mathcal{D}}_{x,t}\langle h\rangle$ (we use h instead of z not to make confusions with [4]) generated by the degree-homogenization of the elements of I. This ideal can be obtained via a standard basis with respect to an order that respects the total degree in the ∂_{x_i}'s. Then we compute a $\prec_L^h$-standard basis G and $G_{|h=1}$ shall be a system of generators of $\mathrm{gr}^L(I)$. This is well known, see for example [6].

In conclusion, we can get $\mathrm{gr}^L(I)$ from I via standard bases computations for (admissible) orders as in [5, 4].

4.2. Here is the proof. For the proof, we shall use tools from [4] and the first part [5].

Consider $b_{L,((f)_{\mathcal{Q}})}$. We don't know a priori whether or not it is zero. By [4, sect. 4–5] applied to the previous algorithm, we have: for a generic $y_0 \in V(\mathcal{Q})$, $b_{L,f(x,y_0)}$ is constant and equal to $b_{L,((f)_{\mathcal{Q}})}$. So the latter is not zero and has rational roots. (Notice that this argument is valid for any $p \geq 2$).

Now consider the ideal I in $\mathcal{D}_{Z \times \mathbf{C}^p/Y}$ generated by the $t_j - f_j$, $j = 1, \dots, p$, and the $\partial_{x_i} + \sum_{j=1}^p (\partial f_j/\partial x_i)\partial_{t_j}$, $i = 1, \dots, n$. From [5], we know that the analytic Gröbner fan of I is constant for a generic $y_0 \in V(\mathcal{Q})$, so that the same is true for its restriction to the space $\sum_{j=1}^p \mathbf{R}_{\geq 0} V_j$, and it equals $\mathcal{E}_V(h((I)_{\mathcal{Q}}))$. Moreover, it follows from Remark 3.2 (it is here that we need to assume $p = 2$) that the element κ obtained from the Gröbner fan (as it is explained in this remark) is generically constant. This κ satisfies property $\mathcal{P}$ for $f(x, y_0)$ for a generic $y_0 \in V(\mathcal{Q})$. This implies that the polynomial

$$\prod_{L \in Sq(\mathcal{E}_V(h((I_{\mathcal{Q}}))))} \prod_{-L(\kappa + (1, \cdots, 1)) < k \leq 0} b_{L,(f)_{\mathcal{Q}}}(L(s) - k)$$

belongs to $\mathcal{B}(f(x, y_0))$ for a generic $y_0 \in V(\mathcal{Q})$. We then apply (iii) in Th. 1 to conclude.

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