Classification of the mapping class groups up to measure equivalence

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Abstract: We study the mapping class groups of compact orientable surfaces from the viewpoint of measure equivalence. In this paper, we announce some classification result of the mapping class groups in terms of measure equivalence and the result that there exist various kinds of discrete groups which are not measure equivalent to the mapping class groups. As a byproduct of the proof, it turns out that the mapping class group is exact.

Key words: The mapping class group; the curve complex; measure equivalence; orbit equivalence.

1. Introduction. Quasi-isometry among finitely generated groups is one of the most fundamental notions in geometric group theory. Gromov gave the following condition equivalent to quasi-isometry between two finitely generated groups (see [12, 0.2.C_2'], [23, Theorem 2.14]). Let Γ and Λ be two finitely generated groups. Then they are quasi-isometric if and only if they are *topologically equivalent* in the following sense: there exist commuting, continuous actions of Γ and Λ on some locally compact space such that the action of each of the groups is properly discontinuous and cocompact.

Inspired by this fact, Gromov introduced the following measure-theoretical counterpart, called measure equivalence:

Definition 1 ([12, 0.5.E]). Two discrete groups Γ and Λ are said to be *measure equivalent* if there exist commuting, measure-preserving, essentially free actions of Γ and Λ on some standard Borel space (Ω, m) with a σ-finite non-zero positive measure such that the action of each of the groups Γ and Λ admits a fundamental domain with finite measure.

In fact, this defines an equivalence relation among discrete groups [6, §2]. Note that two isomorphic groups modulo finite kernels or cokernels are measure equivalent. In this case, we say that the two groups are *almost isomorphic*. It is easy to see that all finite groups form one complete class of measure equivalent groups.

Any two lattices (i.e., discrete subgroups with cofinite volume) in the same locally compact second countable group are measure equivalent [6, Example 1.2]. This example is one of the geometric motivations for introducing the notion of measure equivalence.

It is a fundamental problem to decide whether given two discrete groups are measure equivalent or not. Given a connected compact orientable surface M which may have non-empty boundary, we define the *mapping class group* as the group consisting of all isotopy classes of orientation-preserving self-diffeomorphisms of M. In this paper, we study the mapping class groups from the viewpoint of measure equivalence. More specifically, we give a nearly complete classification result of the mapping class groups in terms of measure equivalence and give many examples of discrete groups not measure equivalent to the mapping class groups. We discuss the exactness of the mapping class groups too.

The purpose of this paper is to announce results, whose proofs and detailed accounts will be published elsewhere [17].

2. Background. We recall some deep results on measure equivalence. Measure equivalence has another equivalent formulation in terms of orbit equivalence. The notion of orbit equivalence has been studied in the framework of ergodic theory and operator algebras for a long time (see [6, 7, 8] for this equivalent formulation and orbit equivalence).

It is infinite amenable groups that were treated first from the viewpoint of orbit equivalence. Thanks to Ornstein-Weiss' result [22], we know that a discrete group is measure equivalent to the infinite

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cyclic group $\mathbf{Z}$ if and only if it is infinite amenable. In the case of non-amenable groups, Zimmer [25] initiated the study of lattices Γ in connected semisimple Lie groups G with finite center and no compact factors and of higher $\mathbf{R}$-rank from the viewpoint of orbit equivalence, and Furman [6] concluded that any discrete group measure equivalent to such Γ is almost isomorphic to a lattice in G.

Gaboriau [9] showed that the ℓ^2-*Betti numbers* for discrete groups, which are introduced by Cheeger-Gromov [5], are invariant under measure equivalence in the following sense: if two discrete groups Γ_1 and Γ_2 are measure equivalent, then there exists a positive constant c such that $\beta_n(\Gamma_1) = c\beta_n(\Gamma_2)$ for all n, where $\beta_n(\Lambda)$ denotes the n-th ℓ^2-Betti number of a discrete group Λ. Thanks to this fact, we can obtain various results about measure equivalence (see [9]).

The reference [10] is a quite detailed survey about recent progress of the study of measure equivalence.

3. Classification. Let $M = M_{g,p}$ be a connected compact orientable surface of *type* (g, p), that is, of genus g and with p boundary components. (Throughout the paper, we assume a surface to be connected, compact and orientable.) Let $\Gamma(M)$ be the mapping class group of M. We set $\kappa(M) = 3g + p - 4$, which is called the *complexity* of M, and set

$$g_0(M) = \begin{cases} 2 & \text{if } g \leq 2, \\ g & \text{if } g > 2. \end{cases}$$

Theorem 1 ([17, Theorem 1.1])**.** *Suppose that M^1 and M^2 are two surfaces with $\kappa(M^1), \kappa(M^2) \geq 0$ and that the mapping class groups $\Gamma(M^1)$ and $\Gamma(M^2)$ are measure equivalent. Then $\kappa(M^1) = \kappa(M^2)$ and $g_0(M^1) = g_0(M^2)$.*

If a surface M satisfies $\kappa(M) < 0$ and is not the torus, then $\Gamma(M)$ is finite. It is known that all $\Gamma(M_{0,4})$, $\Gamma(M_{1,0})$, $\Gamma(M_{1,1})$ and $SL(2, \mathbf{Z})$ are almost isomorphic (see the comment right after [17, Theorem 3.3]), and that $\Gamma(M_{0,6})$ and $\Gamma(M_{2,0})$ are almost isomorphic (see the end of §6.8 in [16]).

The above mentioned fact that the complexity is a measure equivalence invariant can be proved by using the ℓ^2-Betti number as well, though our proof is completely different. The ℓ^2-Betti numbers of the mapping class groups can be calculated by the results due to Gromov [11] and McMullen [20] (see [17, Appendix C]).

4. Discrete groups not measure equivalent to the mapping class groups.

Theorem 2 ([17, Theorem 1.3])**.** *Let M be a surface with $\kappa(M) \geq 0$ and $\Gamma(M)$ the mapping class group of M.*

(i) *If Γ_1 and Γ_2 are two infinite discrete groups and either Γ_1 or Γ_2 has an infinite amenable subgroup, then the direct product $\Gamma_1 \times \Gamma_2$ and a sufficiently large subgroup of $\Gamma(M)$ are not measure equivalent.*

(ii) *If a discrete group Γ has an infinite amenable normal subgroup, then Γ and a sufficiently large subgroup of $\Gamma(M)$ are not measure equivalent.*

A subgroup of $\Gamma(M)$ is said to be *sufficiently large* if it contains an independent pair of pseudo-Anosov elements. A pair of pseudo-Anosov elements is said to be *independent* if their fixed point sets on the Thurston boundary are disjoint. We refer the reader to [17, Theorem 3.1] for the details of sufficiently large subgroups of $\Gamma(M)$.

Adams' method [2] shows Theorem 2 for non-elementary hyperbolic groups instead of $\Gamma(M)$. In fact, the mapping class group has many similar properties to those of non-elementary hyperbolic groups. We shall recall the definition of the curve complex, which is shown to be connected, (Gromov-) hyperbolic and have infinite diameter by Masur-Minsky [18].

Definition 2. Let M be a surface with $\kappa(M) > 0$. The *curve complex* $C = C(M)$ of M is the simplicial complex whose vertex set $V(C)$ is the set of non-trivial isotopy classes of non-peripheral simple closed curves on M, and a (finite) subset of $V(C)$ forms a simplex of C if the set of curves representing it can be realized disjointly on M.

Remark that the curve complex can be defined also for surfaces M with $\kappa(M) = 0$ in a slightly different way (see [17, Definition 2.6]). This complex was introduced by Harvey [14]. By definition, $\Gamma(M)$ acts on C simplicially and thus, isometrically when C is equipped with the natural combinatorial metric. In the proof of theorems in this paper, we use not only hyperbolicity of C, but also special geometric properties of certain geodesics, called *tight* ones, on C, which are discovered by Masur-Minsky [19] and Bowditch [4]. Thanks to the special properties, we can treat (the 1-skeleton of) C like a locally finite hyperbolic graph. For instance, given any two vertices in C, we see that the set of tight geodesics connecting the two vertices is finite.

Although the mapping class group and non-elementary hyperbolic groups have several common properties, we can show the following

Theorem 3 ([17, Theorem 1.6]). *The mapping class group of a surface M with* $\kappa(M) > 0$ *is not measure equivalent to any hyperbolic group.*

Let us denote

$$n(M) = g + \left[\frac{g+p-2}{2}\right]$$

for a surface M of type (g, p), where $[a]$ denotes the maximal integer less than or equal to a for $a \in \mathbf{R}$.

Theorem 4 ([17, Theorem 1.2]). *Let M be a surface with* $\kappa(M) \geq 0$. *Suppose that an infinite subgroup* Γ *of the mapping class group* $\Gamma(M)$ *is measure equivalent to a discrete group containing a subgroup isomorphic to the direct product of n free groups of rank* 2. *Then* $n \leq n(M)$.

In fact, $\Gamma(M)$ itself contains the direct product of $n(M)$ free groups of rank 2 as a subgroup. Therefore, the inequality in Theorem 4 is sharp. We note that there are a few related results counting the number of factors of direct product of certain groups from the viewpoint of measure equivalence as in Theorem 4 (see [9, 21]).

5. Boundary amenability and exactness. One of Adams' important observations is the amenability in a topological sense of the action of a hyperbolic group on its boundary at infinity [1]. We refer the reader to [17, Appendix A] for the definition of the amenability of an action of a discrete group. Since we follow Adams' method, the following similar result need to be shown:

Theorem 5 ([17, Theorem 1.4]). *Let M be a surface with* $\kappa(M) \geq 0$ *and* $\Gamma(M)$ *the mapping class group of M. Let* ∂C *and* $\mathcal{PMF}$ *denote the boundary at infinity of the curve complex and the Thurston boundary of M, respectively. Then*

(i) *the Borel space* ∂C *is standard.*

(ii) *the action of* $\Gamma(M)$ *on* $(\partial C, \mu)$ *is amenable in a measurable sense for any quasi-invariant probability measure* μ *on* ∂C.

(iii) *the action of* $\Gamma(M)$ *on* $(\mathcal{PMF}, \mu)$ *is amenable in a measurable sense for any quasi-invariant probability measure* μ *on* $\mathcal{PMF}$ *with* $\mu(\mathcal{MIN}) = 1$, *where* $\mathcal{MIN}$ *denotes the subset of* $\mathcal{PMF}$ *consisting of all minimal measured foliations.*

It is significant to investigate the amenability of the action of a given discrete group on a (compact) space not only in the study of measure equivalence as above. Let us explain another significance of it. In general, given a continuous action of a discrete group G on a compact Hausdorff space X, we know the following fact [3]: the action is amenable in a topological sense if and only if the action of G on (X, μ) is amenable in a measurable sense for any quasi-invariant measure μ on X. If a discrete group G admits such an amenable action on some compact Hausdorff space X, then we say that G is *exact*. If G is finitely generated, then this property is described as some geometric condition on the Cayley graph of G, which is called the *property A* [15]. The property A for metric spaces was introduced by Yu [24] in his work on the Baum-Connes conjecture. Thanks to his result, the Novikov conjecture is true for all finitely generated groups with the property A.

Although we can see immediately that ∂C is not compact (see [17, Proposition 3.8]) and that the action of $\Gamma(M)$ on $\mathcal{PMF}$ is not amenable in a topological sense since there exist non-amenable stabilizers of the action, we obtain the following

Theorem 6 ([17, Theorem 1.5]). *Let M be a surface with* $\kappa(M) \geq 0$. *Then*

(i) *the curve complex of M has the property A as a metric space.*

(ii) *the mapping class group of M is exact.*

We note that recently, Hamenstädt [13] also shows independently that the mapping class groups are exact by constructing a certain compact Hausdorff space on which it acts amenably in a topological sense.

References

[1] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups, Topology **33** (1994), no. 4, 765–783.

[2] S. Adams, Indecomposability of equivalence relations generated by word hyperbolic groups, Topology **33** (1994), no. 4, 785–798.

[3] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Enseignement Math., Geneva, 2000.

[4] B. H. Bowditch, Tight geodesics in the curve complexes. (Preprint). `http://www.maths.soton.ac.uk/staff/Bowditch`.

[5] J. Cheeger and M. Gromov, L_2-cohomology and group cohomology, Topology **25** (1986), no. 2, 189–215.

[6] A. Furman, Gromov's measure equivalence and rigidity of higher rank lattices, Ann. of Math. (2) **150** (1999), no. 3, 1059–1081.

[7] A. Furman, Orbit equivalence rigidity, Ann. of Math. (2) **150** (1999), no. 3, 1083–1108.

[8] D. Gaboriau, On orbit equivalence of measure preserving actions, in *Rigidity in dynamics and geometry* (*Cambridge*, 2000), 167–186, Springer, Berlin, 2002.

[9] D. Gaboriau, Invariants l^2 de relations d'équivalence et de groupes, Publ. Math. Inst. Hautes Études Sci. No. 95 (2002), 93–150.

[10] D. Gaboriau, Examples of groups that are measure equivalent to the free group, Ergodic Theory Dynam. Systems. (to appear). `math.DS/0503181`.

[11] M. Gromov, Kähler hyperbolicity and L_2-Hodge theory, J. Differential Geom. **33** (1991), no. 1, 263–292.

[12] M. Gromov, Asymptotic invariants of infinite groups, in *Geometric group theory*, *Vol.* 2 (*Sussex*, 1991), 1–295, Cambridge Univ. Press, Cambridge, 1993.

[13] U. Hamenstädt, Geometry of the mapping class groups I: Boundary amenability. (Preprint). `math.GR/0510116`.

[14] W. J. Harvey, Boundary structure of the modular group, in *Riemann surfaces and related topics*: *Proceedings of the* 1978 *Stony Brook Conference* (*State Univ. New York*, *Stony Brook*, *N.Y.*, 1978), 245–251, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[15] N. Higson and J. Roe, Amenable group actions and the Novikov conjecture, J. Reine Angew. Math. **519** (2000), 143–153.

[16] N. V. Ivanov, Mapping class groups, in *Handbook of geometric topology*, 523–633, North-Holland, Amsterdam, 2002.

[17] Y. Kida, The mapping class group from the viewpoint of measure equivalence theory. (Preprint). `http://www.math.kyoto-u.ac.jp/preprint/preprint2005.html`.

[18] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, Invent. Math. **138** (1999), no. 1, 103–149.

[19] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. II. Hierarchical structure, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.

[20] C. T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, Ann. of Math. (2) **151** (2000), no. 1, 327–357.

[21] N. Monod and Y. Shalon, Orbit equivalence rigidity and bounded cohomology, Ann. of Math. (to appear). `http://www.math.uchicago.edu/~monod`.

[22] D. S. Ornstein and B. Weiss, Ergodic theory of amenable group actions. I. The Rohlin lemma, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 161–164.

[23] R. Sauer, L^2-invariants of groups and discrete measured groupoids, Dissertation, Universität Münster (2003).

[24] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. **139** (2000), no. 1, 201–240.

[25] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Basel, 1984.