Gröbner fan for analytic D-modules with parameters

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Abstract: This is the first part of a work dedicated to the study of Bernstein-Sato polynomials for several analytic functions depending on parameters. The main result of this part is a constructibility result for the analytic Gröbner fan of a parametric ideal in the ring of analytic differential operators. In this part, the main tool is the notion of reduced generic standard basis.

Key words: Gröbner fan; standard bases; differential operators.

Introduction. This is the first part of a work dedicated to the study of Bernstein-Sato polynomials of several variables for analytic functions depending on parameters (part 2 is [8]). In the present paper, we focus on a major step of this study and which constitutes a result interesting by its own. It is a constructibility result for the analytic Gröbner fan of a parametric ideal in the ring of analytic differential operators.

The Gröbner fan for polynomials was introduced by Mora and Robbiano in 1988 [14] (but earlier works by Lejeune-Jalabert and Teissier [12] already contain analogous constructions). Since then the Gröbner fan has found numerous applications: e.g. the Gröbner walk in commutative algebra. The "D-modules version" has been treated by Assi et al. [2, 3] and Saito *et al.* [16]. The algebraic version [2, 16] has a nice application to GKZ-hypergeometric differential systems (see [16, Chapters 2, 3]). The analytic version [3] (see also [9] for a significant extension) made possible a complete proof of the existence of Bernstein-Sato polynomials for *several analytic* functions (Bahloul [6]).

Based on [6] our goal is to give new constructive results concerning Bernstein-Sato polynomials for several analytic functions depending on parameters (see the second part of this work [8]). A major step towards this goal consists in studying the Gröbner fan of an ideal depending on parameters in the ring of analytic differential operators. As the main tool for this study, we shall use parametric standard bases for analytic differential ideals (in fact we shall work in a formal setting). Parametric Gröbner bases (for polynomials) have been extensively studied. A local version also exists: see e.g. Greuel and Pfister [11] and Aschenbrenner [1]. An algebraic differential version has been initiated by Oaku [15] whose work inspired e.g. Leykin [13]. The analytic (or formal) differential version has been introduced in Bahloul [7] without using reduced standard bases. In the present paper we add as a supplementary tool the use of parametric standard bases which are reduced in a sense we shall define later. Indeed, since the Gröbner fan is described using reduced standard bases, this is necessary.

Let us summarize: In the first section we introduce some notations and recall some facts about the Gröbner fan, the division theorem and (reduced) standard bases. Section 2 contains the main tool: reduced generic standard bases. Section 3 is devoted to the proof of the main result which we state now:

Theorem 1. *Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be two sets of variables. Let I be an ideal in $\mathbf{C}\{x, y\}[\partial_{x_1}, \dots, \partial_{x_n}]$ the ring of analytic differential operators with parameters y (it is also described as the germ of ring of relative differential operators $\mathcal{D}_{\mathbf{C}^{n+m}/\mathbf{C}^m}$). Let $\mathcal{Q}$ be a prime ideal in $\mathbf{C}\{y\}$. There exists $h(y) \in \mathbf{C}\{y\} \smallsetminus \mathcal{Q}$ such that for any $y_0 \in V(\mathcal{Q})$ with $h(y_0) \neq 0$, the analytic Gröbner fan of $I_{|y=y_0}$ is constant. Here $V(\mathcal{Q})$ means the zero set of a representative of $\mathcal{Q}$ on a small polydisc.*

Remark. We have a similar result concerning the global Gröbner fan for an ideal in $D[y]$, where D is the Weyl algebra over a field of characteristic 0. The proof is easier because every process is finite. One can find a proof in [4, Chap. 6].

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Corollary 2. *There exists a finite stratification of* $(\mathbf{C}^m, 0) = \bigcup W$ *made of locally closed subsets such that the analytic Gröbner fan of* $I_{|y}$ *is constant along each member* W *of the partition.*

Let $\mathcal{E}$ be the common refinement of all the Gröbner fans of $I_{|y=y_0}$, $y_0 \in (\mathbf{C}^m, 0)$ (by Cor. 2, there is only a finite number of such fans). Then $\mathcal{E}$ is the smallest fan with the following property: For any w in an open cone of $\mathcal{E}$ and $y_0 \in (\mathbf{C}^m, 0)$, the graded ideal $\mathrm{gr}^w(I_{|y=y_0})$ is constant. We call $\mathcal{E}$ the *comprehensive Gröbner fan* of I (following the terminology relative to Gröbner bases).

The idea of introducing this fan is due to N. Takayama whom I thank.

1. Recalls: Gröbner fan, divisions. In this section, we recall the homogenized ring of (formal) differential operators $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ ($\mathbf{k}$ is a field with characteristic 0). We then recall the definition of the Gröbner fan as in [3, 9]. Finally we recall the division theorem in $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ together with the notion of (reduced) standard basis (see loc. cit.).

Let $x = (x_1, \ldots, x_n)$ be a system of variables, we write $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ for the corresponding partial differentials. Let z be another variable. The ring $\hat{\mathcal{D}}_n(\mathbf{k})$ is the ring of differential operators with coefficients in $\mathbf{k}[[x]]$. As we will work with arbitrary orders, we will need to work in a homogenized (or graded) version of $\hat{\mathcal{D}}_n(\mathbf{k})$. The ring $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ is defined as the $\mathbf{k}[[x]]$-algebra generated by ∂_x and z where the only non trivial relations are:

$[\partial_{x_i}, a] = (\partial a/\partial x_i)z$ for $i = 1, \ldots, n$ and $a \in \mathbf{k}[[x]]$.

When we replace $\mathbf{k}[[x]]$ with $\mathbf{C}\{x\}$ we obtain and write $\mathcal{D}_n\langle z\rangle$. We see that the previous relation preserves the total degree in the ∂_{x_i}'s and z, which makes $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ a graded algebra for this degree. Note that it is isomorphic to the Rees algebra of $\hat{\mathcal{D}}_n$ associated with the filtration by the degree.

Given $P \in \hat{\mathcal{D}}_n(\mathbf{k})$, $P = \sum c_{\alpha\beta} x^\alpha \partial_x^\beta$ ($\alpha, \beta \in \mathbf{N}^n$, $c_{\alpha\beta} \in \mathbf{k}$), we define $h(P) \in \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ as $h(P) = \sum c_{\alpha\beta} x^\alpha \partial_x^\beta z^{d-|\beta|}$ where $d = \deg(P)$ is the degree of P in the ∂_{x_i}'s. Given a (left) ideal $I \subset \hat{\mathcal{D}}_n(\mathbf{k})$, we define $h(I)$ as the ideal of $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ generated by $\{h(P) \mid P \in I\}$.

Let $w = (u, v) \in \mathbf{Z}^{n+n}$. We consider it as a weight vector on the variables (x, ∂_x). We define $\mathcal{W} = \{w \mid \forall i,\ u_i \leq 0,\ u_i + v_i \geq 0\}$ as the set of *admissible weight vectors*. With $w \in \mathcal{W}$, we can associate a natural filtration on $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ and a graded ring $\mathrm{gr}^w(\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle)$. Let I be a given ideal in $\hat{\mathcal{D}}_n(\mathbf{k})$ (an analoguous construction holds for $\mathcal{D}_n$). For $w, w' \in \mathcal{W}$, we write $w \sim w'$ when $\mathrm{gr}^w(\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle) = \mathrm{gr}^{w'}(\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle)$ and $\mathrm{gr}^w(h(I)) = \mathrm{gr}^{w'}(h(I))$. The partition of $\mathcal{W}$ given by this relation is called the (formal) *open Gröbner fan of* I and denoted by $\mathcal{E}(h(I))$. It is a finite collection of open convex polyhedral cones (see Assi *et al.* [3]). Denote by $\bar{\mathcal{E}}(h(I))$ the set of closures of these open cones and call it the closed Gröbner fan of I. In Bahloul, Takayama [9], we proved $\bar{\mathcal{E}}(h(I))$ is a polyhedral fan improving results of Assi *et al.* [3].

It is easy to prove that the analytic Gröbner fan of $I \subset \mathcal{D}_n$ coincides with the (formal) Gröbner fan of $\hat{\mathcal{D}}_n(\mathbf{C})I$ (see [3, 9]). That is why we shall mainly work in a formal setting.

Let us now deal with divisions and standard bases. A total monomial order $\prec$ on $\mathbf{N}^{2n}$ (or equivalently on the monomials $x^\alpha \xi^\beta$, ξ_i being a comutative variable corresponding to ∂_{x_i}) is said to be *admissible* if $x_i \prec 1$ and $x_i\xi_i \succ 1$. For a given weight vector w, we can define the order $\prec_w$ by refining w by $\prec$ (i.e. we first use w and then tie-break with $\prec$). Obviously if w and $\prec$ are admissible then so is $\prec_w$. Given $\prec$ admissible, we may define $\prec^h$ on $\mathbf{N}^{2n+1}$: $(\alpha, \beta, k) \prec^h (\alpha', \beta', k')$ if $|\beta| + k < |\beta'| + k'$ or ($|\beta| + k = |\beta'| + k'$ and $(\alpha, \beta) \prec (\alpha', \beta')$).

Let $P = \sum c_{\alpha\beta k} x^\alpha \partial_x^\beta z^k$ be in $\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$, we define the support $\mathrm{Supp}(P) \subset \mathbf{N}^{2n+1}$ as the set of (α, β, k) with $c_{\alpha\beta k} \neq 0$. When $P \neq 0$ we define its leading exponent $\exp_{\prec^h}(P) = \max_{\prec^h} \mathrm{Supp}(P)$, leading coefficient $\mathrm{lc}_{\prec^h}(P) = c_{\exp_{\prec^h}(P)}$ and leading monomial $\mathrm{lm}_{\prec^h(P)}(P) = \mathrm{lc}_{\prec^h}(P) \cdot (x, \partial_x, z)^{\exp_{\prec^h}(P)}$. From now on, we shall omit the subscript $\prec^h$ when the context is clear. Let us now recall the division theorem.

Let $P_1, \ldots, P_r \in \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ and $\prec$ an admissible order. With $\prec^h$, we define the partition $\Delta_1 \cup \cdots \cup \Delta_r \cup \bar{\Delta}$ of $\mathbf{N}^{2n+1}$ as $\Delta_1 = \exp(P_1) + \mathbf{N}^{2n+1}$ and for $j \geq 2$, $\Delta_j = (\exp(P_j) + \mathbf{N}^{2n+1}) \setminus \bigcup_{k=1}^{j-1} \Delta_k$.

Theorem 1.1 (Division theorem, see [3, Th. 7], [9, Th. 3.1.1]). *For* $P \in \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$, *there exists a unique* $(Q_1, \ldots, Q_r, R) \in (\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle)^{r+1}$ *such that* $P = \sum_j Q_j P_j + R$ *and*

- *for* $j \geq 1$, $Q_j = 0$ *or* $\mathrm{Supp}(Q_j) + \exp(P_j) \subset \Delta_j$,
- $R = 0$ *or* $\mathrm{Supp}(R) \subset \bar{\Delta}$.

R is called the remainder of the division.

As an easy consequence of the formal division process, we have:

Lemma 1.2. *Let $\mathcal{C}$ be a commutative integral ring and $\mathcal{F}$ a field containing $\mathcal{C}$. Let $P, P_1, \ldots, P_r$ be in $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$. Let us consider the division of P by the P_j's in $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ w.r.t. $\prec^h$: $P = \sum Q_j P_j + R$. Then the coefficients of R and of the Q_j's have the following form*:

$$\frac{c}{\prod_{j=1}^r \mathrm{lc}(P_j)^{d_j}} \quad \textit{where} \quad c \in \mathcal{C},\ d_j \in \mathbf{N}.$$

For a (non zero) left ideal $J \subset \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$, we define the set of exponents $\mathrm{Exp}(J) = \{\exp(P) \mid P \in J,\ P \neq 0\}$. A standard basis $G \subset J$ of J (with respect to $\prec^h$) is defined by the relation: $\mathrm{Exp}(J) = \bigcup_{g\in G}(\exp(g) + \mathbf{N}^{2n+1})$. Thanks to the division theorem, we have: G is standard basis iff (for any P, $P \in J \iff R = 0$, where R is the remainder of the division of P by G). By noetherianity of $\mathbf{N}^{2n+1}$ a standard basis always exists (Dickson lemma).

Let us end these preliminaries with the notion of a reduced standard basis.

Definition 1.3. A $\prec^h$ standard basis $G = \{g_1, \ldots, g_r\}$ of $J \subset \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ is said to be

- minimal if for any $F \subset \mathbf{N}^{2n+1}$, we have: $\mathrm{Exp}_{\prec_L^h}(J) = \bigcup_{e\in F}(e + \mathbf{N}^{2n+1}) \Rightarrow \{\exp(g_1), \ldots, \exp(g_r)\} \subseteq F$.
- reduced if it is minimal and if for any $j = 1, \ldots, r$, $\mathrm{lc}(g_j) = 1$ and $(\mathrm{Supp}(g_j) \smallsetminus \exp(g_j)) \subseteq (\mathbf{N}^{2n+1} \smallsetminus \mathrm{Exp}(J))$.

Given an ideal $J \subset \hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ and $\prec$ admissible, a $\prec^h$-reduced standard basis exists and is unique. Let us sketch the existence:

Let G_0 be any standard basis. By removing unnecessary elements we may assume G_0 to be minimal. Set $G_0 = \{g_j; 1 \le j \le r\}$. For any j, divide $g_j - \mathrm{lm}(g_j)$ by G_0 and denote by r_j the remainder. The set $\{(\mathrm{lm}(g_j) + r_j)/\mathrm{lc}(g_j);\ 1 \le j \le r\}$ is then the reduced standard basis of J.

2. (Reduced) generic standard bases. In [7], we gave a definition of generic standard bases. A more general definition is necessary if we want to deal with the property of reduceness. However for the proofs, we shall cite [7], because they are similar. The symbol $\prec$ shall denote an admissible order.

Let $\mathcal{C}$ be a commutative integral unitary ring (not necessarily noetherian) for which we denote by $\mathcal{F}$ the fraction field, by $\mathrm{Spec}(\mathcal{C})$ the spectrum. We assume also that for any $\mathcal{P} \in \mathrm{Spec}(\mathcal{C})$ and $n \in \mathbf{Z}$, if $n \in \mathcal{P}$ then $1 \in \mathcal{P}$. For any ideal $\mathcal{I}$ in $\mathcal{C}$, we denote by $V(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}); \mathcal{I} \subset \mathcal{P}\}$ the zero set defined by $\mathcal{I}$.

For any $\mathcal{P}$ in $\mathrm{Spec}(\mathcal{C})$ and c in $\mathcal{C}$, denote by $[c]_{\mathcal{P}}$ the class of c in $\mathcal{C}/\mathcal{P}$ and by $(c)_{\mathcal{P}}$ this class viewed in the fraction field $\mathcal{F}(\mathcal{P}) = \mathrm{Frac}(\mathcal{C}/\mathcal{P})$. (The condition above on $\mathcal{C}$ ensures that $\mathcal{F}(\mathcal{P})$ has characteristic 0.) The element $(c)_{\mathcal{P}}$ is called the *specialization of c to $\mathcal{P}$*.

We naturally extend these notations to elements in $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$ and we extend $(\,\cdot\,)_{\mathcal{P}}$ to elements of $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ for which the denominators of the coefficients are in $\mathcal{C} \smallsetminus \mathcal{P}$, i.e. $\hat{\mathcal{D}}_n(\mathcal{C}_{\mathcal{P}})\langle z\rangle$ where $\mathcal{C}_{\mathcal{P}}$ is the localization w.r.t. $\mathcal{P}$.

Now, given an ideal $J \subset \hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$, we define the specialization $(J)_{\mathcal{P}}$ of J to $\mathcal{P}$ as the ideal of $\hat{\mathcal{D}}_n(\mathcal{F}(\mathcal{P}))\langle z\rangle$ generated by all the $(P)_{\mathcal{P}}$ with $P \in J$.

2.1. Generic standard basis on an irreducible affine scheme. Fix a prime ideal $\mathcal{Q}$ in $\mathcal{C}$. Let us start with some notations. We denote by $\hat{\mathcal{D}}_n(\mathcal{Q})\langle z\rangle$ the ideal of $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$ made of elements with all their coefficients in $\mathcal{Q}$. For $h \in \mathcal{C}$, we denote by $\mathcal{C}[h^{-1}]$ the localization of $\mathcal{C}$ w.r.t. h. The ring $\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle$ shall be seen as the subring of $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ made of elements with coefficients c/c' such that c' is a power of h. In the latter, if all the c are in $\mathcal{Q}$, we obtain an ideal denoted by $\hat{\mathcal{D}}_n(\mathcal{Q}[h^{-1}])\langle z\rangle$. Finally, $\langle\mathcal{Q}\rangle$ denotes the ideal of $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ made of elements with coefficients c/c' such that $c \in \mathcal{Q}$. Notice that the latter is a priori different from $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle\mathcal{Q}$ since we don't suppose $\mathcal{C}$ to be noetherian.

Now for an element P in $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle \smallsetminus \hat{\mathcal{D}}_n(\mathcal{Q})\langle z\rangle$ or more generally in $\hat{\mathcal{D}}_n(\mathcal{C}_{\mathcal{Q}})\langle z\rangle \smallsetminus \langle\mathcal{Q}\rangle$, let us write $P = \sum(c_{\alpha\beta k}/c'_{\alpha\beta k})x^\alpha\partial_x^\beta z^k$ with $c'_{\alpha\beta k} \in \mathcal{C} \smallsetminus \mathcal{Q}$. Then denote by $\exp^{\mathrm{mod}\mathcal{Q}}(P)$ the maximum (w.r.t. $\prec^h$) of the (α, β, k) with $c_{\alpha\beta k} \notin \mathcal{Q}$. This is the leading exponent of P modulo $\mathcal{Q}$. In the same way, we define the leading coefficient $\mathrm{lc}^{\mathrm{mod}\mathcal{Q}}(P)$ and leading monomial $\mathrm{lm}^{\mathrm{mod}\mathcal{Q}}(P)$ modulo $\mathcal{Q}$.

In general we have $\exp^{\mathrm{mod}\mathcal{Q}}(PQ) = \exp^{\mathrm{mod}\mathcal{Q}}(P) + \exp^{\mathrm{mod}\mathcal{Q}}(Q)$ as for the usual leading exponent. However, there are some differences with the usual situation, for example the leading coefficient mod$\mathcal{Q}$ of PQ is not equal to the product of that of them. They are equal only modulo $\mathcal{Q}$ so we will have to be careful.

Now for an ideal $J \subset \hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle \smallsetminus \hat{\mathcal{D}}_n(\mathcal{Q})\langle z\rangle$, we define: $\mathrm{Exp}^{\mathrm{mod}\mathcal{Q}}(J) = \{\exp^{\mathrm{mod}\mathcal{Q}}(P) \mid P \in J \smallsetminus \hat{\mathcal{D}}_n(\mathcal{Q})\langle z\rangle\}$. This set is stable by sums in $\mathbf{N}^{2n+1}$ thus by Dickson lemma:

$$(1) \quad \begin{cases} \exists\{g_1,\dots,g_r\} \subset J \text{ such that} \\ \mathrm{Exp}^{\mathrm{mod}\mathcal{Q}}(J) = \bigcup_j (\exp^{\mathrm{mod}\mathcal{Q}}(g_j) + \mathbf{N}^n). \end{cases}$$

This shall be a generic standard basis of J on $V(\mathcal{Q})$. However, this is not the definition we will adopt. In fact in the next paragraph we will define the notion of reduced generic standard basis and it will not be in the ring $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$ so we need a more general definition:

Definition 2.1. A generic standard basis (gen.s.b for short) of J on $V(\mathcal{Q})$ is a couple $(\mathcal{G}, h)$ where

(a) $h \in \mathcal{C} \smallsetminus \mathcal{Q}$,
(b) $\mathcal{G}$ is a finite set in the ideal $\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle \cdot J$ and for any $g \in \mathcal{G}$ the numerator of $\mathrm{lc}^{\mathrm{mod}\mathcal{Q}}(g)$ divides h,
(c) $\mathrm{Exp}^{\mathrm{mod}\mathcal{Q}}(J) = \bigcup_{g\in\mathcal{G}} (\exp^{\mathrm{mod}\mathcal{Q}}(g) + \mathbf{N}^{2n+1})$.

Above in (1), $\big(\{g_1,\dots,g_r\}, \prod_j \mathrm{lc}^{\mathrm{mod}\mathcal{Q}}(g_j)\big)$ is a gen.s.b of J on $V(\mathcal{Q})$. Thus, Def. 2.1 makes sens. Notice that another way to state (b) is: For any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h)$, the specialization $(g)_{\mathcal{P}}$ is well defined and belongs to $(J)_{\mathcal{P}}$ and $\exp((g)_{\mathcal{P}})$ is equal to $\exp^{\mathrm{mod}\mathcal{Q}}(g)$. Note that $V(\mathcal{Q}) \smallsetminus V(h)$ is non empty since $h \notin \mathcal{Q}$.

Proposition 2.2 (Division modulo $\mathcal{Q}$). *Let $h \in \mathcal{C} \smallsetminus \mathcal{Q}$ and $g_1,\dots,g_r$ be in $\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle \smallsetminus \hat{\mathcal{D}}_n(\mathcal{Q}[h^{-1}])\langle z\rangle$ such that each $\mathrm{lc}(g_j)$ divides h. Let $\Delta_1 \cup \cdots \cup \Delta_r \cup \bar{\Delta}$ be the partition of $\mathbf{N}^{2n+1}$ associated with the $\exp^{\mathrm{mod}\mathcal{Q}}(g_j)$. Then for any P in $\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle$, there exist $q_1,\dots,q_r,R,T \in \hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ such that*

(o) $P = \sum_j q_j g_j + R + T$,
(i) $\mathrm{Supp}(q_j) + \exp^{\mathrm{mod}\mathcal{Q}}(g_j) \subset \Delta_j$ *if* $q_j \neq 0$,
(ii) $\mathrm{Supp}(R) \subset \bar{\Delta}$ *if* $R \neq 0$,
(iii) *the q_j and R are in $\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle$ and T is in $\hat{\mathcal{D}}_n(\mathcal{Q}[h^{-1}])\langle z\rangle$.*

Moreover, $(q_1,\dots,q_r,R)$ is unique modulo $\hat{\mathcal{D}}_n(\mathcal{Q}[h^{-1}])\langle z\rangle$. We call R the remainder mod$\mathcal{Q}$ *of this division.*

Sketch of proof. Write $g_j = g_j^{(1)} - g_j^{(2)}$ with $g_j^{(2)} \in \langle\mathcal{Q}\rangle$ and $\exp(g_j^{(1)}) = \exp^{\mathrm{mod}\mathcal{Q}}(g_j)$ then divide P by the $g_j^{(1)}$'s in $\hat{\mathcal{D}}_n(\mathcal{F})\langle z\rangle$ as in Theorem 1.1: $P = \sum_j q_j g_j^{(1)} + R$. We have, $P = \sum_j q_j g_j + R + T$ with $T = \sum_j q_j g_j^{(2)}$. Conditions (i) and (ii) are satisfied by Theorem 1.1. The third one is a direct consequence of Lemma 1.2. The last statement comes from the unicity in Th. 1.1 after specializing to $\mathcal{Q}$. □

The main result concerning generic standard basis is the following

Theorem 2.3. *Let $(\mathcal{G}, h)$ be a gen.s.b of $J \subset \hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$ on $V(\mathcal{Q})$. Then for any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h)$:*

(i) $(\mathcal{G})_{\mathcal{P}} \subset (J)_{\mathcal{P}}$,
(ii) $\mathrm{Exp}((J)_{\mathcal{P}}) = \bigcup_{g\in\mathcal{G}} (\exp((g)_{\mathcal{P}}) + \mathbf{N}^n) = \mathrm{Exp}^{\mathrm{mod}\mathcal{Q}}(J)$.

In other words, $(\mathcal{G})_{\mathcal{P}}$ is a standard basis of $(J)_{\mathcal{P}}$ for a generic $\mathcal{P} \in V(\mathcal{Q})$ and $\mathrm{Exp}((J)_{\mathcal{P}})$ is generically constant and equal to $\mathrm{Exp}^{\mathrm{mod}\mathcal{Q}}(J)$.

Proof. Exactly the same as for [7, Th. 3.7] □

2.2. Reduced generic standard bases. The next result shall concern the existence of *the* reduced generic standard basis on $V(\mathcal{Q})$ (in fact we shall see that it is unique "modulo $\mathcal{Q}$").

Let J be an ideal in $\hat{\mathcal{D}}_n(\mathcal{C})\langle z\rangle$ and $\mathcal{Q}$ be a prime ideal in $\mathcal{C}$.

Theorem 2.4 (Definition-Theorem)**.**

- *There exists a gen.s.b $(\mathcal{G}, h)$ of J on $V(\mathcal{Q})$ such that $(\mathcal{G})_{\mathcal{Q}}$ is the reduced standard basis of $(J)_{\mathcal{Q}}$. Such a $(\mathcal{G}, h)$ is called a reduced generic standard basis (red.gen.s.b) of J on $V(\mathcal{Q})$.*
- *If $(\mathcal{G}, h)$ is a red.gen.s.b on $V(\mathcal{Q})$ then for any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h)$, $(\mathcal{G})_{\mathcal{P}}$ is the reduced standard basis of $(J)_{\mathcal{P}}$.*

Such a red.gen.s.b is unique "modulo $\langle\mathcal{Q}\rangle$." More precisely:

Lemma 2.5. *Let $(\mathcal{G}, h)$ and $(\mathcal{G}', h')$ be two red.gen.s.b of J on $V(\mathcal{Q})$ then*

- *their cardinality and the set of their leading exponents* mod$\mathcal{Q}$ *are equal,*
- *if $g \in \mathcal{G}$ and $g' \in \mathcal{G}'$ satisfy $\exp^{\mathrm{mod}\mathcal{Q}}(g) = \exp^{\mathrm{mod}\mathcal{Q}}(g')$ then $g - g'$ belongs to $\hat{\mathcal{D}}_n(\mathcal{Q}[(hh')^{-1}])\langle z\rangle$.*

Proof. The first statement is trivial by unicity of reduced standard bases. For the second one, we have $g - g' \in \hat{\mathcal{D}}_n(\mathcal{C}[hh'^{-1}])\langle z\rangle$ and by the same argument of unicity, $(g)_{\mathcal{Q}} - (g')_{\mathcal{Q}} = 0$ thus $g - g' \in \hat{\mathcal{D}}_n(\mathcal{Q}[hh'^{-1}])\langle z\rangle$. □

Proof of the theorem. For the first statement, let $(\mathcal{G}_0, h)$ be any gen.s.b of J on $V(\mathcal{Q})$. Set $\mathcal{G}_0 = \{g_1,\dots,g_r\}$. By removing the unnecessary elements, we may assume that it is minimal. For any j we may assume $\mathrm{lc}^{\mathrm{mod}\mathcal{Q}}(g_j)$ to be unitary. For any j, let r_j be the remainder mod$\mathcal{Q}$ of the division modulo $\mathcal{Q}$ of $g_j - \mathrm{lm}^{\mathrm{mod}\mathcal{Q}}(g_j)$ by $\mathcal{G}_0$. Set $\mathcal{G} = \big\{\mathrm{lm}^{\mathrm{mod}\mathcal{Q}}(g_j) + r_j \bigm| j = 1,\dots,r\big\}$. It is easy to check that $(\mathcal{G}, h)$ is a red.gen.s.b.

Let us prove the second statement. Let $(\mathcal{G},h)$ be a red.gen.s.b. First, we know that for any $\mathcal{P}\in V(\mathcal{Q})\smallsetminus V(h)$, $(\mathcal{G})_{\mathcal{P}}$ is a standard basis of $(J)_{\mathcal{P}}$. Moreover it is minimal since $\mathrm{Exp}((J)_{\mathcal{P}})=\mathrm{Exp}((J)_{\mathcal{Q}})$ and $\exp((g)_{\mathcal{P}})=\exp^{\mathrm{mod}\,\mathcal{Q}}(g)=\exp((g)_{\mathcal{Q}})$ for any $g\in\mathcal{G}$. The latter also implies that it is unitary. It just remains to prove that it is reduced. But this follows from the fact that $(\mathcal{G})_{\mathcal{Q}}$ is reduced and that for any $g\in\mathcal{G}$, $\mathrm{Supp}((g)_{\mathcal{P}})\subset\mathrm{Supp}((g)_{\mathcal{Q}})$ (since $\mathcal{Q}\subset\mathcal{P}$). □

The following example shows that reduced generic standard bases need to be defined in some extension as in Def. 2.1.

Example 2.6. Take $f(x_1,x_2,y)=yx_2-x_1x_2+x_1$ in $\mathbf{C}[y][[x]]$ (it is a commutative situation for simplification). Take an order such that the leading exponent (in terms of x) is $(0,1)$ (i.e. corresponding to x_2). Take $\mathcal{Q}=(0)$ in $\mathbf{C}[y]$ then the red.gen.s.b of the ideal generated by f is $x_2+x_1/y+x_1^2/y^2+x_1^3/y^3+\cdots$.

One can find more details and more results on (reduced) generic standard bases in Bahloul [5].

3. Back to Gröbner fans.

In this section we shall prove Theorem 1. First let us recall two results needed for the proof.

Take $I\subset\hat{\mathcal{D}}_n(\mathbf{k})$. For $w,w'\in\mathcal{W}$ we have defined an equivalence relation $w\sim w'$ and defined the open Gröbner fan as the collection of the equivalence classes. For $w\in\mathcal{W}$, let us denote by $C_w(h(I))$ the equivalence class of w.

Claim 3.1 (Recall 1). *Let $w'\in C_w(h(I))$ and $\prec$ any admissible order then the reduced standard bases of $h(I)\subset\hat{\mathcal{D}}_n(\mathbf{k})\langle z\rangle$ with respect to $\prec_w^h$ and $\prec_{w'}^h$ coincide.*

This is by construction of the Gröbner fan because $\mathrm{Exp}_{\prec_w^h}(h(I))=\mathrm{Exp}_{\prec_{w'}^h}(h(I))$ (see [3]). Another way to see this equality: By definition $\mathrm{gr}^w(h(I))=\mathrm{gr}^{w'}(h(I))$, so $\mathrm{Exp}_{\prec_w^h}(h(I))=\mathrm{Exp}_{\prec^h}(\mathrm{gr}^w(h(I)))=\mathrm{Exp}_{\prec^h}(\mathrm{gr}^{w'}(h(I)))=\mathrm{Exp}_{\prec_{w'}^h}(h(I))$ (see [9, Lemma 3.2.2]).

For $g\in h(I)$, we define the Newton polyhedron as the following convex hull: $\mathrm{New}(g)=\mathrm{conv}(\mathrm{Supp}(g)+\{(\alpha,\beta,0)\in\mathbf{Z}^{2n+1}\mid\forall(u,v)\in\mathcal{W},\ (u,v)\cdot(\alpha,\beta)\le 0\})$. See [9, 3.3].

Claim 3.2 (Recall 2). *Let $w\in\mathcal{W}$ and $\prec$ be an admissible order. Let G be the $\prec_w^h$-reduced standard basis of $h(I)$. Let $Q=\sum_{g\in G}\mathrm{New}(g)$ be the Minkowski sum of the Newton polyhedra, then*

$$C_w(h(I))=\mathrm{N}_Q(\mathrm{face}_w(Q)).$$

Here $\mathrm{N}_Q(\mathrm{face}_w(Q))$ denotes the normal cone in Q of the face of Q with respect to w.

See Prop. 3.3.3 of [9] for the proof and 3.3 of loc. cit. for more details.

Lemma 3.3. *Suppose we are given $\mathcal{Q}\subset\mathcal{C}$ prime, $h\in\mathcal{C}\smallsetminus\mathcal{Q}$ and $g\in\hat{\mathcal{D}}_n(\mathcal{C}[h^{-1}])\langle z\rangle$. Then there exists $H\in\mathcal{C}\smallsetminus\mathcal{Q}$ such that for any $\mathcal{P}\in V(\mathcal{Q})\smallsetminus V(H)$, $\mathrm{New}((g)_{\mathcal{P}})=\mathrm{New}((g)_{\mathcal{Q}})$.*

Proof. Denote by $\mathcal{W}^\star$ the polar dual cone of $\mathcal{W}$, $\mathcal{W}^\star=\{\sum_{i=1}^n\lambda_ie_i+\lambda_i'e_i'\mid\lambda_i,\lambda_i'\ge 0\}$, where $e_i\in\mathbf{Z}^{2n}$ is the vector having 1 in its ith component and 0 for the others, and $e_i'\in\mathbf{Z}^{2n}$ has -1 at positions i and $n+i$ and 0 at the others. Then there exists a finite subset $E(g)\subset\mathrm{Supp}(g)$ such that $\mathbf{N}ew((g)_{\mathcal{Q}})=\mathrm{conv}(E(g))+\mathcal{W}^\star$ (the argument uses Dickson lemma, see [9, 3.3]).

Let us write $g=\sum_\gamma(c_\gamma/h^{l_\gamma})(x,\partial_x,z)^\gamma$, with $\gamma\in\mathbf{N}^{2n+1}$, $c_\gamma\in\mathcal{C}$, $l_\gamma\in\mathbf{N}$, and set $H=h\prod_{\gamma\in E(g)}c_\gamma(g)$.

Take $\mathcal{P}\in V(\mathcal{Q})\smallsetminus V(H)$ and let us prove $\mathrm{New}((g)_{\mathcal{P}})=\mathrm{New}((g)_{\mathcal{Q}})$.

Since $\mathcal{P}\supset\mathcal{Q}$, we have $\mathrm{Supp}((g)_{\mathcal{P}})\subset\mathrm{Supp}((g)_{\mathcal{Q}})$ thus $\mathrm{New}((g)_{\mathcal{P}})\subset\mathrm{New}((g)_{\mathcal{Q}})$. Let us prove the inverse inclusion.

We have $\mathrm{Supp}((g)_{\mathcal{Q}})\subset\mathrm{conv}(E(g))+\mathcal{W}^\star$ but $E(g)\subset\mathrm{Supp}((g)_{\mathcal{P}})$ (because $H\notin\mathcal{P}$), therefore $\mathrm{Supp}((g)_{\mathcal{Q}})\subset\mathrm{New}((g)_{\mathcal{P}})$. This implies the desired inclusion and completes the proof. □

We are now ready to prove Theorem 1.

Proof. The ideal I is in $\mathbf{C}\{x,y\}[\partial_x]$. We can consider a representative of I and we regard it as an ideal of $\hat{\mathcal{D}}_n(\mathcal{C})$ where $\mathcal{C}=\mathcal{O}_{\mathbf{C}^m}(U)$ and U is a polydisc in $\mathbf{C}^m$. We shall prove that there exists $h\in\mathcal{C}\smallsetminus\mathcal{Q}$ such that for any $\mathcal{P}\in V(\mathcal{Q})\smallsetminus V(h)$ the Gröbner fan $\mathcal{E}(h((I)_{\mathcal{P}}))$ does not depend on $\mathcal{P}$. This will prove our theorem because the formal Gröbner fan and the analytic Gröbner fan coincide.

From now on, we assume I in $\hat{\mathcal{D}}_n(\mathcal{C})$. First let us prove the following

Claim 3.4. *There exists $h'\in\mathcal{C}\smallsetminus\mathcal{Q}$ such that for a $\mathcal{P}$ in $V(\mathcal{Q})\smallsetminus V(h')$, $h((I)_{\mathcal{P}})=(h(I))_{\mathcal{P}}$.*

Consider the order $\prec_t$ defined on $\mathbf{N}^{2n}$ as follows: Take any admissible order $\prec$. Take $t=(0,\ldots,0,1,\ldots,1)$ (the number of 0's and 1's is n) and refine t by $\prec$ to obtain $\prec_t$. Here t stands for "total degree." For such an order we also have a formal division in $\hat{\mathcal{D}}_n(\mathbf{k})$ and the notion of standard basis (see e.g. Castro-Jiménez, Granger [10]). Therefore, we have a notion of generic standard basis (see

Bahloul [7]).

Now take $(\mathcal{G}', h')$ a generic standard basis of I for $\prec_t$. For any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h')$, $(\mathcal{G}')_\mathcal{P}$ is a $\prec_t$-standard basis of $(I)_\mathcal{P}$. Therefore, $h((I)_\mathcal{P})$ is generated by $h((\mathcal{G}')_\mathcal{P})$ (this is well known, see e.g. [9, 4.2]).

Take P in I. Consider the division modulo $\mathcal{Q}$ of P by $\mathcal{G}'$ with $\prec_t$: $P = \sum_g q_g g + R + T$ as in Prop. 2.2. By specializing to $\mathcal{Q}$, $(P)_\mathcal{Q} = \sum_g (q_g)_\mathcal{Q}(g)_\mathcal{Q} + (R)_\mathcal{Q}$, we obtain the division of $(P)_\mathcal{Q}$ by $(\mathcal{G}')_\mathcal{Q}$ as in Theorem 1.1, thus $(R)_\mathcal{Q} = 0$ because $(\mathcal{G}')_\mathcal{Q}$ is a standard basis, i.e. R equals zero modulo $\mathcal{Q}$, we may assume it is zero. By definition of $\prec_t$, $\deg(P) \geq \deg(q_g g)$ and $\deg(P) \geq \deg(T)$, therefore $h(P) = \sum z^{l_g} h(q_g)h(g) + z^l h(T)$ for some integers l and l_g. When specializing to $\mathcal{P}$, we get $(h(P))_\mathcal{P} = \sum z^{l_g}(h(q_g))_\mathcal{P}(h(g))_\mathcal{P}$. We also notice that $(h(g))_\mathcal{P} = h((g)_\mathcal{P})$ (because the leading exponent which has maximum degree is preserved after specialization to $\mathcal{P}$) and that $(h(I))_\mathcal{P}$ is generated by the $(h(P))_\mathcal{P}$, $P \in I$. Thus, we can conclude that $h((I)_\mathcal{P})$ and $(h(I))_\mathcal{P}$ are both generated by $h((\mathcal{G}')_\mathcal{P})$. Claim 3.4 is proven.

Let us go back to the proof: in the sequel we shall use Claim 3.4 without any explicit mention. Consider the (open) Gröbner fan of $(I)_\mathcal{Q}$: $\{C_{w_1}(h((I)_\mathcal{Q})), \ldots, C_{w_s}(h((I)_\mathcal{Q}))\}$. For each i, let $(\mathcal{G}_i, h_i)$ be a red.gen.s.b. of $h(I)$ for $\prec^h_{w_i}$.

Define h'' as the product of all the H's obtained when we apply Lemma 3.3 to all the elements of the $\mathcal{G}_i$'s (this product is finite). Now take $w \in \mathcal{W}$ arbitrary, put $h = h'h''h_1 \cdots h_s$ (h' comes from Claim 3.4) and let us prove that for any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h)$, $C_w(h((I)_\mathcal{P}))$ does not depend on $\mathcal{P}$.

Since $\mathcal{W} = \cup_i C_{w_i}(h((I)_\mathcal{Q}))$, there exists i with $w \in C_{w_i}(h((I)_\mathcal{Q}))$. By definition, $(\mathcal{G}_i)_\mathcal{Q}$ is the reduced standard basis of $h((I)_\mathcal{Q})$ for $\prec^h_{w_i}$. By Recall 1, it is also the reduced standard basis for $\prec^h_w$. Thus by definition, $(\mathcal{G}_i, h_i)$ is a red.gen.s.b of $h(I)$ for $\prec^h_w$. By Theorem 2.4, $(\mathcal{G}_i)_\mathcal{P}$ is the reduced standard basis of $h((I)_\mathcal{P})$ for $\prec^h_w$. By using Recall 2 and lemma 3.3, we can conclude that: $C_w(h((I)_\mathcal{P})) = C_w(h((I)_\mathcal{Q}))$. $\square$

Remark. The progress of the proof shows that Th. 1 is true for other situations (e.g. if the ideal I is in $\mathcal{D}_n[y]$).

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