# Discrete tomography and the Hodge conjecture for certain abelian varieties of CM-type

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**Abstract:** We study a problem in discrete tomography on $\mathbf{Z}^n$, and show that there is an intimate connection between the problem and the study of the Hodge cycles on abelian varieties of CM-type. This connection enables us to apply our results in tomography to obtain several infinite families of abelian varieties for which the Hodge conjecture holds.

**Key words:** Discrete tomography; Hodge cycle.

**1. Introduction.** Tomography usually reconstructs the shape of a solid object from images of successive plane sections of it. (The so-called "CT" is an abbreviation of "Computerized Tomography.") Discrete tomography, on the other hand, tries to reconstruct a function $f$ on $\mathbf{Z}^n$ from various sums $f_{\mathbf{t}+\mathbf{v}} = \sum_{\mathbf{x}\in\mathbf{t}+\mathbf{v}} f(\mathbf{x})$, $\mathbf{v} \in \mathbf{Z}^n$, where $\mathbf{t}$ (called a *window*) is a fixed finite subset of $\mathbf{Z}^n$. (See section two for precise definition.) The purpose of this paper is to anounce several results in discrete tomography by arbitrary windows in $\mathbf{Z}^n$, and show that it leads us naturally to the proof of the Hodge conjecture for certain abelian varieties with complex multiplication by abelian CM-fields.

The plan of this paper is as follows: In Section two, we define the object of our main concern, the space $\mathbf{A}^0_\mathbf{t}$ of bounded arrays of degree zero with respect to every translation of a window $\mathbf{t}$, and formulate the basic problems. Section three gives a dimension formula for $\mathbf{A}^0_\mathbf{t}$, and Section four concerns with the periodicity of arrays in $\mathbf{A}^0_\mathbf{t}$, and describes a criterion for $\mathbf{A}^0_\mathbf{t}$ to contain a multiply periodic array. Section five examines several examples and shows how to apply the general results to investigate concrete examples of windows. In Section six we reveal an intimate connection between discrete tomography and the study of the Hodge rings of abelian varieties with complex multiplication by abelian CM-field. Details will appear elsewhere.

**2. Problem setting.** In this section we introduce some notation and formulate the basic problems of our concern.

Let $\mathbf{A} = (\mathbf{C})^{\mathbf{Z}^n}$ denote the set of $\mathbf{C}$-valued functions on $\mathbf{Z}^n$. We write its element in the form $\mathbf{a} = (\mathbf{a_i})$ where $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{Z}^n$ and $\mathbf{a_i} \in \mathbf{C}$. We call an element of $\mathbf{A}$ simply an *array.* When there exists a positive constant $C$ such that $|\mathbf{a_i}| < C$ for any $\mathbf{i} \in \mathbf{Z}^n$, the array is said to be *bounded.* We denote the set of bounded arrays by $\mathbf{A}^0$. For any array $\mathbf{a} = (\mathbf{a_i})$, let $\operatorname{supp}\mathbf{a} = \{\mathbf{i} \in \mathbf{Z}^n\,;\, \mathbf{a_i} \neq 0\} \subset \mathbf{Z}^n$ and call it the *support* of $\mathbf{a}$. An array with finite support is called a *window*, and the set of windows is denoted by $\mathbf{W}$. For any window $\mathbf{t} = (\mathbf{t_i})$ and for any array $\mathbf{a} = (\mathbf{a_i})$, let $d_\mathbf{t}(\mathbf{a}) = \sum_{\mathbf{i}\in\mathbf{Z}^n} \mathbf{t_i}\mathbf{a_i}$ and call it the *degree of* $\mathbf{a}$ *with respect to* $\mathbf{t}$. Furthermore let

$$\mathbf{A}^0_\mathbf{t} = \left\{\mathbf{a} \in \mathbf{A}^0\,;\, d_{\mathbf{t}+\mathbf{p}}(\mathbf{a}) = 0 \quad \textit{for any} \quad \mathbf{p} \in \mathbf{Z}^n\right\},$$

the set of bounded arrays of degree zero with respect to every translation of $\mathbf{t}$. Here the translated window $\mathbf{t}+\mathbf{p}$ is defined by $(\mathbf{t}+\mathbf{p})_\mathbf{i} = \mathbf{t}_{\mathbf{i}-\mathbf{p}}$, $\mathbf{i} \in \mathbf{Z}^n$. The main problems we want to study in this paper are the following ones:

(2.a) *to find a condition for finite-dimensionality of* $\mathbf{A}^0_\mathbf{t}$,

(2.b) *to find an explicit formula for the dimension of* $\mathbf{A}^0_\mathbf{t}$,

(2.c) *to find a condition under which* $\mathbf{A}^0_\mathbf{t}$ *contains a multiply periodic array.*

**3. Dimension formula for $\mathbf{A}^0_\mathbf{t}$.** In this section we solve the problems (2.a) and (2.b).

In order to formulate our result, we introduce some notation. For any window $\mathbf{t} = (\mathbf{t_i}) \in \mathbf{W}$, let $m_\mathbf{t}(\mathbf{z}) = \sum_{\mathbf{i}\in\mathbf{Z}^n} \mathbf{t_i}\mathbf{z^i} \in \mathbf{C}\left[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}\right]$, where $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{z^i} = z_1^{i_1} \cdots z_n^{i_n}$. We

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call it the *characteristic polynomial* of $\mathbf{t}$. Let $\mathbf{T} = \{z \in \mathbf{C}\,;\ |z| = 1\}$ and let $\iota\colon \mathbf{T}^n \to \mathbf{T}^n$ denote the automorphism of $\mathbf{T}^n$ defined by $\iota(z_1,\dots,z_n) = (z_1^{-1},\dots,z_n^{-1})$. We put $m_{\mathbf{t}}^* = \iota^*(m_{\mathbf{t}})$ so that $m_{\mathbf{t}}^*(\mathbf{z}) = m_{\mathbf{t}}(\iota(\mathbf{z}))$. For any subset $X \subset \mathbf{C}^n$, we denote the zero locus $\{\mathbf{z} \in X\,;\, m_{\mathbf{t}}(\mathbf{z}) = 0\}$ by $V_X(m_{\mathbf{t}})$.

**Theorem 3.1.** *Suppose that $V_{\mathbf{T}^n}(m_{\mathbf{t}}^*)$ is a finite set. Then we have*

$$\dim_{\mathbf{C}} A_{\mathbf{t}}^0 = \#\left(V_{\mathbf{T}^n}(m_{\mathbf{t}}^*)\right).$$

*When $V_{\mathbf{T}^n}(m_{\mathbf{t}}^*)$ is infinite, the space $\mathbf{A}_{\mathbf{t}}^0$ is infinite-dimensional.*

For the proof we employ the theory of pseudomeasures on the $n$-dimensional torus and their Fourier transforms, as is derscribed in [1].

**4. Periodicity of arrays in $\mathbf{A}_{\mathbf{t}}^0$.** In this section we solve the problem (2.c).

Let $\mu_n \subset \mathbf{T}$ denote the set of the $n$-th roots of unity and let $\mu_\infty = \bigcup_{n\geq 1} \mu_n$. An array $\mathbf{a} = (\mathbf{a}_{\mathbf{i}})_{\mathbf{i}\in\mathbf{Z}^n} \in \mathbf{A}$ is said to be *$n$-ply periodic*, if there exists an element $\mathbf{c} = (c_1,\dots,c_n) \in \mathbf{Z}_{\geq 1}^n$ such that $\mathbf{a}_{\mathbf{i}} = \mathbf{a}_{\mathbf{i}+\mathbf{c}}$ holds for any $\mathbf{i} \in \mathbf{Z}^n$. The following theorem provides us with a criterion for periodicity:

**Theorem 4.1.** *For any window $\mathbf{t}$, there exists a nonzero $n$-ply periodic array in $\mathbf{A}_{\mathbf{t}}^0$ if and only if $V_{\mu_\infty^n}(m_{\mathbf{t}}) \neq \emptyset$.*

**Remark.** The condition $V_{\mu_\infty^n}(m_{\mathbf{t}}) \neq \emptyset$ is equivalent to $V_{\mu_\infty^n}(m_{\mathbf{t}}^*) \neq \emptyset$, since $\iota$ restricts to a bijection on $\mu_\infty^n$.

**Corollary 4.1.1.** *For any window $\mathbf{t}$, there exists a nonzero $n$-ply periodic array with period $(c_1,\dots,c_n)$ in $\mathbf{A}_{\mathbf{t}}^0$ if and only if $V_{\mu_{c_1}\times\cdots\times\mu_{c_n}}(m_{\mathbf{t}}) \neq \emptyset$.*

**5. Applications.** In this section, we apply Theorem 3.1 and Theorem 4.1 to determine the structure of $\mathbf{A}_{\mathbf{t}}^0$ for some examples of 2-dimensional windows.

**5.1. Window $\mathbf{t}_{\mathbf{harmonic}}$.** This is defined by

$$(\mathbf{t}_{\text{harmonic}})_{(i,j)} = \begin{cases} -4, & \text{if} \quad (i,j) = 0, \\ 1, & \text{if} \quad |i| + |j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(See Remark 5.1.1 (1) below for the reason why we call it *harmonic*.) The characteristic polynomial is given by $m_{\mathbf{t}_{\text{harmonic}}} = w + (z - 4 + z^{-1}) + w^{-1}$. Let $(z_0, w_0) \in V_{\mathbf{T}^2}\bigl(m_{\mathbf{t}_{\text{harmonic}}}^*\bigr)$. Then we have

$$\begin{aligned} m_{\mathbf{t}_{\text{harmonic}}}^*(z_0,w_0) &= m_{\mathbf{t}_{\text{harmonic}}}\bigl(z_0^{-1}, w_0^{-1}\bigr) \\ &= w_0^{-1} + \bigl(z_0^{-1} - 4 + z_0\bigr) + w_0 = 0, \end{aligned}$$

and hence $w_0 + z_0 + z_0^{-1} + w_0^{-1} = 4$. This is possible only if $z_0 = w_0 = 1$, since $(z_0,w_0) \in \mathbf{T}^2$. Hence we see that $V_{\mathbf{T}^2}\bigl(m_{\mathbf{t}_{\text{harmonic}}}^*\bigr) = \{(1,1)\}$ and $\dim \mathbf{A}_{\mathbf{t}_{\text{harmonic}}}^0 = \#\bigl(V_{\mathbf{T}^2}\bigl(m_{\mathbf{t}_{\text{harmonic}}}^*\bigr)\bigr) = 1$ by Theorem 3.1. On the other hand, it is clear that the all-one array $\mathbf{1}$ belongs to $\mathbf{A}_{\mathbf{t}_{\text{harmonic}}}^0$. Hence we obtain the following

**Proposition 5.1.** *For the window $\mathbf{t}_{\text{harmonic}}$, we have $\mathbf{A}_{\mathbf{t}_{\text{harmonic}}}^0 = \{c.\mathbf{1}\,;\, c \in \mathbf{C}\}$.*

**Remark 5.1.1.** (1) Note that an array $\mathbf{a} = \bigl(\mathbf{a}_{(i,j)}\bigr)_{(i,j)\in\mathbf{Z}}$ belongs to $\mathbf{A}_{\mathbf{t}_{\text{harmonic}}}^0$ if and only if it is bounded and $\mathbf{a}_{(i,j)} = \bigl(\mathbf{a}_{(i+1,j)} + \mathbf{a}_{(i,j+1)} + \mathbf{a}_{(i-1,j)} + \mathbf{a}_{(i,j-1)}\bigr)/4$ for any $(i,j) \in \mathbf{Z}^2$. Hence it gives rise to a *discrete harmonic function* on the lattice $\mathbf{Z}^2$. Thus Proposition 5.1 says that any bounded discrete harmonic function on $\mathbf{Z}^2$ must be constant.
(2) One can generalize the proposition to the $n$-dimensional window $\mathbf{t}_{\text{harmonic}}^n$ defined by

$$(\mathbf{t}_{\text{harmonic}}^n)_{\mathbf{i}} = \begin{cases} -2^n, & \text{if} \quad \mathbf{i} = 0, \\ 1, & \text{if} \quad \sum_{1\leq j\leq n} |i_j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus any bounded discrete harmonic function on $\mathbf{Z}^n$ must be constant.

**5.2. Window $\mathbf{t}_{\mathbf{stairs}}(N)$ $(N \geq 1)$.** We define $\mathbf{t}_{\text{stairs}}(N) = \bigl(\mathbf{t}_{(i,j)}\bigr)$ by

$$\mathbf{t}_{(i,j)} = \begin{cases} 1, & \text{if} \quad 0 \leq i, j, i+j \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 5.2.** *Let $R_n^* = \mu_n - \{1\}$, the set of nontrivial $n$-th roots of unity, and let $\Delta_n$ denote the diagonal of $R_n^* \times R_n^*$. Then we have*

$$\begin{aligned} V_{\mathbf{T}^2}\bigl(m_{\mathbf{t}_{\text{stairs}(N)}}\bigr) &= (R_{N+1}^* \times R_{N+1}^* - \Delta_{N+1}) \\ &\quad \cup (R_{N+2}^* \times R_{N+2}^* - \Delta_{N+2}). \end{aligned}$$

Note that the right hand side is stable under $\iota\colon \mathbf{T}^2 \to \mathbf{T}^2$, hence we see that

(5.1)
$$\begin{aligned} V_{\mathbf{T}^2}\bigl(m_{\mathbf{t}_{\text{stairs}(N)}}^*\bigr) &= (R_{N+1}^* \times R_{N+1}^* - \Delta_{N+1}) \\ &\quad \cup (R_{N+2}^* \times R_{N+2}^* - \Delta_{N+2}) \end{aligned}$$

holds too. Since $\#(R_n^* \times R_n^* - \Delta_n) = (n-1)^2 - (n-1) = (n-1)(n-2)$ for any $n$ and $(R_{N+1}^* \times R_{N+1}^* - \Delta_{N+1}) \cap (R_{N+2}^* \times R_{N+2}^* - \Delta_{N+2}) = \emptyset$, the equality (5.1) together with Theorem 3.1 and Theorem 4.1 implies the following corollaries.

**Corollary 5.2.1.** $\dim A_{\mathbf{t}_{\text{stairs}(N)}}^0 = 2N^2$.

**Corollary 5.2.2.** *Every array in* $A^0_{\mathbf{t}_{\text{stairs}}(N)}$ *is doubly periodic with period* $(\operatorname{lcm}(N+1, N+2), \operatorname{lcm}(N+1, N+2))$.

**5.3. Window $\mathbf{t}_{\mathrm{leg}}(a,b)$.** This is defined by

$$(\mathbf{t}_{\text{leg}}(a,b))_{(i,j)} = \begin{cases} 1, & \text{if} \quad 0 \le i \le a \quad \text{and} \quad j = 0, \\ & \text{or if} \quad i = 0 \quad \text{and} \quad 0 \le j \le b, \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 5.3.** $V_{\mathbf{T}^2}\big(m_{\mathbf{t}_{\text{leg}}(a,b)}\big) = (R_a^* \times R_{b+1}^*) \cup (R_{a+1}^* \times R_b^*) \cup \Delta'_{a+b+1}$, *where* $\Delta'_{a+b+1} = \{(z,w) \in R_{a+b+1}^* \times R_{a+b+1}^*\,;\, zw = 1\}$.

This implies the following corollaries through Theorems 3.1 and 4.1.

**Corollary 5.3.1.** *For any pair* $(a,b)$ *of positive integers, we have*

$$\dim \mathbf{A}^0_{\mathbf{t}_{\text{leg}}(a,b)} = 2(ab+1) - (a, b+1) - (a+1, b).$$

**Corollary 5.3.2.** *Every array in* $\mathbf{A}^0_{\mathbf{t}_{\text{leg}}(a,b)}$ *is doubly periodic.*

**6. Application to the study of Hodge cycles.** In this section we show that our results in discrete tomography play an important role in the study of the ring of Hodge cycles on certain abelian varieties of CM-type.

For any $n$-tuple $\mathbf{c} = (c_1, \ldots, c_n)$ of integers $\ge 2$, we consider a CM-field $K_{\mathbf{c}}$ such that the Galois group $G(K_{\mathbf{c}}/\mathbf{Q})$ is isomorphic to the abelian group $G_{\mathbf{c}} = \mathbf{Z}/2\mathbf{Z} \times H_{\mathbf{c}}$, where $H_{\mathbf{c}} = \prod_{1 \le j \le n} \mathbf{Z}/c_j\mathbf{Z}$, and the complex conjugation $\rho$ corresponds to $(1, 0, \ldots, 0) \in G_{\mathbf{c}}$. A subset $T \subset G_{\mathbf{c}}$ is called a *CM-type* if

$$G_{\mathbf{c}} = T \amalg \rho(T) \quad \text{(disjoint sum)}.$$

Let $\mathbf{G}_{\mathbf{c}} = \mathbf{Z}[G_{\mathbf{c}}]$ and let $\mathbf{H}_{\mathbf{c}} = \mathbf{Z}[H_{\mathbf{c}}]$, the latter being regarded as a subring of $\mathbf{G}_{\mathbf{c}}$ through the natural inclusion map. Furthermore we put

$$\mathbf{G}_{\mathbf{c}}^{\ge 0} = \left\{ \sum_{g \in G_{\mathbf{c}}} c_g.g \in \mathbf{G}_{\mathbf{c}}\,;\, c_g \ge 0 \quad \text{for any} \quad g \in G_{\mathbf{c}} \right\}.$$

We will write the group operation on $G_{\mathbf{c}}$ multiplicatively in order to tell it from the addition in the group ring. Through this convention any element $g_1 \in G_{\mathbf{c}}$ acts as an automorphism of $\mathbf{G}_{\mathbf{c}}$ by the rule $g_1\left(\sum_{g \in G_{\mathbf{c}}} c_g.g\right) = \sum_{g \in G_{\mathbf{c}}} c_g.g_1 g$. Let $p\colon \mathbf{G}_{\mathbf{c}} \to \mathbf{H}_{\mathbf{c}}$ denote the projection defined by $p\left(\sum_{g \in G_{\mathbf{c}}} c_g.g\right) = \sum_{g \in H_{\mathbf{c}}} c_g.g$. For any subset $S \subset G_{\mathbf{c}}$ let $[S] = \sum_{s \in S} s \in \mathbf{G}_{\mathbf{c}}$. We define a linear map $\varphi\colon \mathbf{G}_{\mathbf{c}} \to \mathbf{H}_{\mathbf{c}}$ by

$$\varphi(\mathbf{v}) = p(\mathbf{v} - \rho\mathbf{v}), \quad \mathbf{v} \in \mathbf{G}_{\mathbf{c}}.$$

Let

$$\left(\mathbf{G}_{\mathbf{c}}^{\ge 0}\right)_{\text{nondiv}} = \left\{ \sum_{g \in G_{\mathbf{c}}} c_g.g \in \mathbf{G}_{\mathbf{c}}^{\ge 0}\,;\, c_g c_{\rho g} = 0 \quad \text{for any} \quad g \in G_{\mathrm{c}} \right\}$$

(We will see below that each element of $\left(\mathbf{G}_{\mathbf{c}}^{\ge 0}\right)_{\text{nondiv}}$ gives rise to a nondivisorial Hodge cycle on a certain abelian variety constructed from these data.) We introduce a natural $\mathbf{Z}$-valued pairing $\langle\ ,\ \rangle_{\mathbf{G}_{\mathbf{c}}}$ by $\left\langle \sum_{g \in G_{\mathbf{c}}} c_g.g, \sum_{g \in G_{\mathbf{c}}} d_g.g \right\rangle_{\mathbf{G}_{\mathbf{c}}} = \sum_{g \in G_{\mathbf{c}}} c_g d_g$.

We recall below some facts on Hodge cycles on abelian varieties of CM-type (see [3] for details). Let $T \subset G_{\mathbf{c}}$ be a CM-type and let $A_T$ denote the abelian variety associated to $T$. One knows that the first cohomology group $H^1(A_T, \mathbf{C})$ can be identified with $\mathbf{C}^{G_{\mathbf{c}}}$, and the complexification of the Hodge ring $(\subset \bigwedge(\mathbf{C}^{G_{\mathbf{c}}}))$ admits as basis the set of basis vector of $\bigwedge(\mathbf{C}^{G_{\mathbf{c}}})$ corresponding to subsets $P$ of $G_{\mathbf{c}}$ with the property that

$$\#(P \cap gT) = (\#P)/2 \quad \text{for any} \quad g \in G_{\mathbf{c}}.$$

This condition can be reformulated in terms of the group algebra $\mathbf{G}_{\mathbf{c}}$ as

$$\langle [P], [gT] - \rho[gT] \rangle_{\mathbf{G}_{\mathbf{c}}} = 0 \quad \text{for any} \quad g \in G_{\mathbf{c}}.$$

We can generalize the above consideration to deal with the Hodge ring of $A_T^N = A_T \times \cdots \times A_T$ ($N$ times), by using the isomorphism $H^1\big(A_T^N, \mathbf{C}\big) \cong (\mathbf{C}^{G_{\mathbf{c}}})^{\oplus N}$. For any $i \in [1, N]$, let $e_g^i$, $g \in G_{\mathbf{c}}$, denote the standard basis of the $i$-th direct summand of $(\mathbf{C}^{G_{\mathbf{c}}})^{\oplus N}$. For any $\mathbf{v} = \sum_{g \in G_{\mathbf{c}}} c_g.g \in \mathbf{G}_{\mathbf{c}}^{\ge 0}$ with $c_g \le N$, we denote by $\langle \mathbf{v} \rangle$ the basis element of $\bigwedge\big((\mathbf{C}^{G_{\mathbf{c}}})^{\oplus N}\big)$ defined by

$$\langle \mathbf{v} \rangle = \bigwedge_{g \in G_{\mathbf{c}}} \left( \bigwedge_{1 \le i_g \le c_g} e_g^{i_g} \right).$$

We have seen in [3] that $\langle \mathbf{v} \rangle$ is a Hodge cycle on $A_T^N$ if and only if $\langle \mathbf{v}, [gT] - \rho[gT] \rangle_{G_{\mathbf{c}}} = 0$ for any $g \in G_{\mathbf{c}}$. Furthermore, when $A_T$ is simple, one knows that $\langle \mathbf{v} \rangle$ is nondivisorial if and only if $c_g c_{\rho g} = 0$ holds for any $g \in G_{\mathbf{c}}$. In view of this, we put

$$\mathbf{G}_{\mathbf{c}}(T)_{\text{Hodge}} = \left\{ \mathbf{v} \in \mathbf{G}_{\mathbf{c}}^{\ge 0}; \langle \mathbf{v}, [gT] - \rho[gT] \rangle_{\mathbf{G}_{\mathbf{c}}} = 0 \text{ for any } g \in G_{\mathbf{c}} \right\},$$

$$\mathbf{G}_{\mathbf{c}}(T)_{\text{Hodge,nondiv}} = \mathbf{G}_{\mathbf{c}}(T)_{\text{Hodge}} \cap \left(\mathbf{G}_{\mathbf{c}}^{\ge 0}\right)_{\text{nondiv}}.$$

A CM-type $T \subset G_{\mathbf{c}}$ is said to be *primitive* if the corresponding abelian variety $A_T$ is simple. By [4], $T$ is primitive if and only if there exists no $g \in G_c - \{\mathbf{0}\}$ such that $g.T = T$. We summarize the above argument in the following form:

**Proposition 6.1.** *For any* $\mathbf{v} = \sum_{g \in G_{\mathbf{c}}} c_g.g \in \mathbf{G}_{\mathbf{c}^{\geq 0}}$,

(i) $\langle \mathbf{v} \rangle$ *is a Hodge cycle on some self-product of* $A_T$ *if and only if* $\mathbf{v} \in \mathbf{G}_{\mathbf{c}}(T)_{\mathrm{Hodge}}$.

(ii) *When* $T$ *is primitive*, $\langle \mathbf{v} \rangle$ *is a nondivisorial Hodge cycle on some self-product of* $A_T$ *if and only if* $\mathbf{v} \in \mathbf{G}_{\mathbf{c}}(T)_{\mathrm{Hodge,nondiv}}$.

We will see that the sets $\mathbf{G}_{\mathbf{c}}(T)_{\mathrm{Hodge}}$ and $\mathbf{G}_{\mathbf{c}}(T)_{\mathrm{Hodge,nondiv}}$ are related with a certain set of arrays investigated in the previous sections. For any element $\mathbf{w} = \sum_{h \in H_{\mathbf{c}}} d_h.h$ of $\mathbf{H}_{\mathbf{c}}$, we define a window $\mathbf{t}^{\mathbf{w}} = (\mathbf{t}^{\mathbf{w}}_{\mathbf{i}})_{\mathbf{i} \in \mathbf{Z}^n}$ by the rule

$$\mathbf{t}^{\mathbf{w}}_{\mathbf{i}} = \begin{cases} d_{\pi_{\mathbf{c}}(\mathbf{i})}, & \mathbf{i} \in [\mathbf{0}, \mathbf{c} - \mathbf{1}], \\ 0, & \text{otherwise}. \end{cases}$$

where $\pi_{\mathbf{c}} \colon \mathbf{Z}^n \to H_{\mathbf{c}}$ denotes the natural projection. We denote by $\mathbf{A}(\mathbf{Z})$ the set of $\mathbf{Z}$-valued arrays, and let $\mathbf{A}^0_{\mathbf{t}}(\mathbf{Z}) = \mathbf{A}^0_{\mathbf{t}} \cap \mathbf{A}(\mathbf{Z})$. In this notation, $\pi_{\mathbf{c}}$ induces an injective homomorphism $\pi^*_{\mathbf{c}} \colon \mathbf{H}_{\mathbf{c}} \, (= (\mathbf{Z})^{H_{\mathbf{c}}}) \to \mathbf{A}(\mathbf{Z}) \, (= (\mathbf{Z})^{\mathbf{Z}^n})$, whose image coincides with

$$\mathbf{A}(\mathbf{Z})^{\mathbf{c}} = \{(\mathbf{a}_{\mathbf{i}})_{\mathbf{i} \in \mathbf{Z}^n} \in \mathbf{A}(\mathbf{Z})\, ; \mathbf{a}_{\mathbf{i}+\mathbf{c}} = \mathbf{a}_{\mathbf{i}} \text{ for any } \mathbf{i} \in \mathbf{Z}^n\},$$

the set of $n$-ply periodic arrays with period $\mathbf{c}$.

**Theorem 6.2.** *Let* $T \subset G_{\mathbf{c}}$ *be a CM-type. For an element* $\mathbf{v} \in \mathbf{G}^{\geq 0}_{\mathbf{c}}$ *to belong to* $\mathbf{G}_{\mathbf{c}}(T)_{\mathrm{Hodge}}$, *it is necessary and sufficient that* $\pi^*_{\mathbf{c}}(\varphi(\mathbf{v})) \in \mathbf{A}^0_{\mathbf{t}^{\varphi([T])}}(\mathbf{Z})^{\mathbf{c}}$.

In view of Proposition 6.1, Theorem 6.2 enables us to relate the study of Hodge cycles with that of discrete tomography in the following form:

**Theorem 6.3.** *Let* $T \subset G_{\mathbf{c}}$ *be a CM-type and let* $\mathbf{v} \in \mathbf{G}^{\geq 0}_{\mathbf{c}}$. *Then* $\langle \mathbf{v} \rangle$ *is a Hodge cycle on some self-product of the abelian variety* $A_T$ *if and only if* $\pi^*_{\mathbf{c}}(\varphi(\mathbf{v})) \in \mathbf{A}^0_{\mathbf{t}^{\varphi[T]}}(\mathbf{Z})^{\mathbf{c}}$.

Next we will study Hodge rings of an infinite family of abelian varieties constructed from a fixed finite subset of $\mathbf{Z}^n_{\geq 0}$. For any finite subset $S \subset \mathbf{Z}^n_{\geq 0}$, let $\mathrm{Rec}(S) = \{\mathbf{c} \in \mathbf{Z}^n_{\geq 2}\, ; [\mathbf{0}, \mathbf{c} - \mathbf{1}] \supset S\}$, where $\mathbf{Z}_{\geq n}$ denotes the set of integers greater than or equal to $n$. When $\mathbf{c} \in \mathrm{Rec}(S)$, we regard $S$ as a subset of $H_{\mathbf{c}}$ through the natural projection. Let

$$\begin{aligned} &\mathrm{Rec}(S)_{\mathrm{nonprim}} \\ &= \{\mathbf{c} \in \mathrm{Rec}(S)\, ; \\ &\qquad h.S = S \text{ or } S' \text{ for some } h \in H_{\mathbf{c}} - \{(0, \ldots, 0)\}\}, \\ &\mathrm{Rec}(S)_{\mathrm{prim}} = \mathrm{Rec}(S) - \mathrm{Rec}(S)_{\mathrm{nonprim}}, \end{aligned}$$

where $S'$ denotes the complement of $S \subset H_{\mathbf{c}}$ in $H_{\mathbf{c}}$. We denote by $\mathrm{AV}(S)$ the set of abelian varieties $A_{T_{\mathbf{c},S}}$, $\mathbf{c} \in \mathrm{Rec}(S)$, with complex multiplication by $K_{\mathbf{c}}$ such that its CM-type $T_{\mathbf{c},S}$ is given by

$$(6.5) \qquad T_{\mathbf{c},S} = (\{0\} \times S) \cup (\{1\} \times (S')) \subset G_{\mathbf{c}}.$$

It follows from [4] that $A_{\mathbf{c}}$ is simple if and only if $\mathbf{c} \in \mathrm{Rec}(S)_{\mathrm{prim}}$. Recall that an abelian variety $A$ of CM-type is said to be *stably nondegenerate* if there are no nondivisorial Hodge cycles on $A$ as well as any of its self-products [2]. If $A$ is not stably nondegenerate, then it is said to be *stably degenerate*. The following theorem determines completely which abelian varieties in $\mathrm{AV}(S)$ are simple and stably nondegenerate.

**Theorem 6.4.** *Notation being as above, let*

$$\mathrm{Period}(S) = \left\{\mathbf{c} \in \mathrm{Rec}(S)\, ; V_{\mu_{c_1} \times \cdots \times \mu_{c_n}}\left(m_{\mathbf{t}[S]}\right) \neq \emptyset\right\}.$$

*Then we have*

$$\begin{aligned} &\{\mathbf{c} \in \mathrm{Rec}(S)\, ; A_{T_{\mathbf{c},S}} \textit{ is simple and stably nondegenerate}\} \\ &= \mathrm{Rec}(S)_{\mathrm{prim}} - \mathrm{Period}(S). \end{aligned}$$

We examine how this theorem contributes to the study of Hodge cycles through several examples. First we deal with the window $\mathbf{t}_{\mathrm{stairs}}(N)$ treated in 5.2.

**Example 6.5.** Let $S = \mathbf{t}_{\mathrm{stairs}}(N)$, $N \geq 1$. In this case we have

$$\mathrm{Rec}(\mathbf{t}_{\mathrm{stairs}}(N))_{\mathrm{prim}} = \mathrm{Rec}(\mathbf{t}_{\mathrm{stairs}}(N)) = \mathbf{Z}^2_{\geq N+1},$$

Furthermore Proposition 5.2 tells us that

$$\mathrm{Period}(\mathbf{t}_{\mathrm{stairs}}(N)) = \mathbf{Z}^2_{\geq N+1} \cap (\mathrm{Pair}_{N+1} \cup \mathrm{Pair}_{N+2}),$$

where we put $\mathrm{Pair}_n = \{(a,b) \in \mathbf{Z}^2_{\geq 0}\, ; (a,n), (b,n) > 1\} - \{(a,b) \in \mathbf{Z}^2_{\geq 0}\, ; (a,n) = (b,n) = 2\}$ for any $n$. (Note that $\mathrm{Pair}_2 = \emptyset$ by definition.) Thus it follows from Theorem 6.4 that for any positive $N$ and for any $\mathbf{c} \in \mathbf{Z}^2_{\geq N+1} - (\mathrm{Pair}_{N+1} \cup \mathrm{Pair}_{N+2})$, the abelian variety $A_{T_{\mathbf{c},\mathbf{t}_{\mathrm{stairs}}(N)}}$ is simple and stably nondegenerate. In particular, we see that

*there exist infinitely many stably nondegenerate abelian varieties in* $\mathrm{AV}(\mathbf{t}_{\mathrm{stairs}}(N))$, *and hence Hodge conjecture holds for infinitely many abelian varieties in* $\mathrm{AV}(\mathbf{t}_{\mathrm{stairs}}(N))$.

Moreover the same theorem implies also that

*there exist infinitely many stably degenerate abelian varieties in* $\mathrm{AV}(\mathbf{t}_{\mathrm{stairs}}(N))$.

The following two examples examine the simplest and the second simplest $n$-dimensional windows.

**Proposition 6.6.** *Let* $\mathbf{O} = \{(0,\dots,0)\} \subset \mathbf{Z}^n$. *Then any abelian varieties in* $\mathrm{AV}(\mathbf{O})$ *are simple and stably nondegenerate. In particular the Hodge conjecture holds for every abelian variety in* $\mathrm{AV}(\mathbf{O})$.

**Corollary 6.6.1.** *Let $K$ be an arbitrary abelian CM-field which contains an imaginary quadratic subfield. Then there exists at least one CM-type for $K$ such that the corresponding abelian variety satisfies the Hodge conjecture.*

**Remark.** Actually, anyone with a little experience of computing Hodge cycles on abelian varieties could prove Proposition 6.6 directly without any knowledge about discrete tomography. The point is, however, that discrete tomography leads us naturally to the simplest window, which gives rise, a posteriori, to infinitely many stably nondegenerate abelian varieties as above.

The next example deals with the second simplest window. The result is, however, rather different. For any integer $n$, let $\mathbf{Z}_{\mathrm{even},\geq n}$ (resp. $\mathbf{Z}_{\mathrm{odd},\geq n}$) denote the set of even (resp. odd) integers $\geq n$.

**Proposition 6.7.** *Let* **Domino** *denote the subset* $\{\mathbf{0}, \mathbf{e}_1\} \subset \mathbf{Z}^n$, *where* $\mathbf{e}_1 = \{1,0,\dots,0\}$. *Then we have*

$$\mathrm{Rec}(\mathbf{Domino})_{\mathrm{nonprim}} = \{2\} \times \mathbf{Z}_{\geq 2}^{n-1},$$
$$\mathrm{Rec}(\mathbf{Domino})_{\mathrm{prim}} = \mathbf{Z}_{\geq 3} \times \mathbf{Z}_{\geq 2}^{n-1}.$$

*Furthermore every abelian variety* $A_{T_{\mathbf{c},\mathbf{Domino}}}$ *with* $\mathbf{c} \in \mathbf{Z}_{\mathrm{odd},\geq 3} \times \mathbf{Z}_{\geq 2}^{n-1}$ *is stably nondegenerate. On the other hand, every abelian variety* $A_{T_{\mathbf{c},\mathbf{Domino}}}$ *with* $\mathbf{c} \in \mathbf{Z}_{\mathrm{even},\geq 4} \times \mathbf{Z}_{\geq 2}^{n-1}$ *is stably degenerate.*

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