Meromorphic solutions of functional equations with nonconstant coefficients

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Abstract: We have continued, by utilizing Nevanlinna's value distribution theory, our previous studies on the existence or solvability of meromorphic solutions of functional equations with constant coefficients to that of similar types of functional equations with meromorphic (small functions) coefficients. The results obtained are relating to value sharing or unicity of meromorphic functions.

Key words: value distribution theory; functional equation; admissible solution.

1. Introduction and results. The well-known Picard's theorem states that if, on the complex plan $\mathbf{C}$, a meromorphic function f fails to take three values on $\overline{\mathbf{C}}$, the extended complex plane, then f must be a constant. Thus, for instance, the equation: $e^z = a$, for any value a other than $0, \infty$, must have at least one root in $\mathbf{C}$. However, Picard's theorem won't be able to tell us how many such roots? In 1920s, R. Nevanlinna [10] developed the so called value distribution theory which enables one to give a relatively elementary proof of the Picard's theorem and, more importantly, provides the above equation a quantitative estimation of the number of the roots in terms of the growth of the function f. Since then the studies of value distribution theory has been extended to algebroid functions, meromorphic mappings, in terms of the concepts of Riemann surfaces and tools such Ahlfor's covering theorem, complex geometry or differential geometry. We refer the reader to ref [5] and [7] for the basic notations and theorems of the so called Nevanlinna's value distribution theory. The theory mainly consists of two so called main theorems; first and second fundamental theorems. We recall here that the two fundamental theorems are valid if the values a_is are replaced by $a_i(z)$s, the so called small functions of f which means each of the $a_i(z)$s satisfying: $T(r, a_i) = S(r, f)$. As an immediate application of the second fundamental theorem, Nevanlinna himself established the following

Theorem A (Five-value Theorem). *If f and g are two nonconstant meromorphic functions and $a_1, a_2, \dots, a_5$ be five distinct values on $\overline{\mathbf{C}}$ such that $f^{-1}(a_i) = g^{-1}(a_i)$ IM (ignoring multiplicity), for $i = 1, 2, \dots, 5$, then $f \equiv g$.*

In 1982, Gross-Yang [3] extended the above result by studying the preimage sets of a finite number of sets each consisting of a finite number of distinct values and the concept of the so called unique range set; a set S is called unique range set of entire functions or URSE, if whenever $f^{-1}(S) = g^{-1}(S)$ CM (counting multiplicity) for two nonconstant entire functions f and g, then $f \equiv g$, and URSM is deinfed similarly for meromorphic functions. It's easily shown that there does exist such a set S with infinitely many elements. Gross [2] asked whether or not there exists a finite set which is URS? In 1995, Yi [14] exhibited such sets of entire functions and then, by Li-Yang [8], of meromorphic functions. As any finite set of distinct values can be expressed as roots of a polynomials (or a rational function with only one simple pole, if S includes ∞). Without loss of generality, we may always assume $S = \{z \mid P(z) = 0, \text{for some polynomial } P \text{ with simple zeros}\}$. Thus if two functions f and g share ∞ and S CM, then we have the functional equation: $P(f) = P(g)e^{\alpha}$, for some entire function α. Thus if the corresponding set S is URS, then one can derive from the equation, mainly from value distribution theory, that α is a constant and in fact $e^{\alpha} = 1$. Then from the equation: $P(f) = P(g)$, one can conclude that $f \equiv g$ and no other nonconstant solutions f and

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g. In [12] Yang-Hua exhibits some classes of such polynomials, including the polynomials of the form: $P(z) = z^n + az^{n-m} + b$, where a, b are nonzero constants and m, n are positive integers relatively prime to each other and $n \geq 5$. Such a polynomial is called unique polynomial of meromorphic functions or "UPM", i.e., whenever $P(f) = P(g)$ for two nonconstant meromorphic functions f and g, then $f = g$. Thus any unique range set of meromorphic functions will lead to a corresponding unique polynomial, but the converse is not true in general. In [8], it has been shown that any polynomial of degree not exceed 4 is not an UPM. On the other hand, value distribution theory has been used to study the Fermat type of equations of meromorphic functions since 1960s (see e.g. [1, 11]). And we refer the reader to [4] for some recent developments of value sharing and more general Fermat type of equations of meromorphic functions and to [6] for that of meromorphic mappings.

In the note, we shall extend our previous studies and results to Diophantine type of equations which have polynomials or, more generally, meromorphic functions as the coefficients. An admissible (meromorphic) solution f of such an equation means the condition $T(r, a_i) = S(r, f)$ holds for all the coefficients appeared in the equations. In the sequel, for a given equation, we shall only consider the existence or non-existence of its admissible (meromorphic) solutions. Essentially, based on the techniques developed by us and peers, particularly those in [8], we are able to establish the following results.

Theorem 1. *Let a_i, b_i, $(i = 1, 2)$ and c be meromorphic functions none of them is identically zero; Let $n, m\,(\geq 2)$ be positive integers and relatively prime to each other, and $n > 2m + 3$. Then the following equation:*

$$(1) \qquad f^n + a_1 f^{n-m} + b_1 = c\left(g^n + a_2 g^{n-m} + b_2\right)$$

has a pair of admissible solution (f, g), iff $c = b_1/b_2$ and $f = hg$, where h is meromorphic function satisfying $h^n = c$ and $h^m = a_1/a_2$.

Corollary 1. *Given integers, m and n with $n > 2m + 3$ $(m \geq 2)$, relatively prime to each other, and rational functions a_1, a_2, a_3 and a_4 $(\not\equiv 0)$, the following functional equation:*

$$(2) \qquad f^n + a_1 f^{n-m} + a_2 g^n + a_3 g^{n-m} + a_4 = 0$$

has no transcendental meromorphic solution f and g.

Theorem 2. *Let a_1, a_2, a_3 be meromorphic functions and $a_1 \not\equiv 0$ or $a_3 \not\equiv 0$. If the positive integer triple (n, m, k) satisfies $k > 1$, $m < n$, and $n > k(m + 2)/(k - 1)$ or $n < k(m - 2)$, then the following equation:*

$$(3) \qquad f^n + a_1 f^{n-m} + a_2 g^k = a_3$$

has no pair of admissible meromorphic solution (f, g).

Theorem 3. *Suppose that a is a nonconstant meromorphic function, and $P(z)$ a polynomial of degree n of the form*

$$(4) \quad P(z) = c_0 + c_1(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where c_0, c_1 $(c_0 c_1 \neq 0)$ and $z_j\,(j = 1, \ldots, k)$ $(z_i \neq z_j$, whenever $i \neq j)$ are complex numbers, and $n > 2k + 1$. Then there exist no meromorphic functions f and g with a being their small function, and

$$(5) \qquad P(f) = aP(g).$$

Remark. The condition $n > 2k + 1$ in Theorem 3 is necessary. For example, if $P(z) = -1 + z^2$, then the functions

$$f = \frac{e^{2z} - 2ae^z + a}{a - e^{2z}}, \quad g = \frac{e^{2z} - 2e^z + a}{e^{2z} - a}$$

satisfy equation (5) for any nonconstant rational function a. In fact, we have the following

Theorem 4. *Suppose, in (5), that $a(z)$ is a nonconstant meromorphic function, $P(z)$ a polynomial of degree n of the form*

$$(6) \qquad P(z) = (z - z_1)^{n-1}(z - z_2),$$

where z_1 and z_2 are distinct complex numbers, then any pair of solution of equation (5) can be expressed as

$$f = z_1 + \frac{(z_2 - z_1)h(a - h^{n-1})}{a - h^n},$$
$$g = z_1 + \frac{(z_2 - z_1)(a - h^{n-1})}{a - h^n},$$

where h is arbitrary meromorphic function such that $a(z)$ is small function of h.

If the small function $a(z)$ in (5) is replaced by ae^{α}, then we have the following

Theorem 5. *Suppose that $a(z)$ is a nonconstant meromorphic function, $P(z)$ a polynomial of degree n with expression of (4). If $k > 1$ and $n > 4k + 2$ then for any entire function α, the equation*

$$(7) \qquad P(f) = ae^{\alpha}P(g).$$

has no admissible meromorphic solutions f and g such that a is a small function with respect to f and g.

We refer the reader to [9] which is to be published elsewhere, for the detailed proofs of the results. However, as an illustration of the arguments or proofs of these theorems, we, in this note, shall only present a detailed proof of Theorem 1.

2. Proof of Theorem 1. First of all, we prove the following

Lemma 1. *Suppose that f is a nonconstant meromorphic function and n, m are positive integers relatively prime to each other. Suppose that $a \not\equiv 0$, $b \not\equiv 0$ are small functions with respect to f. If $\frac{f^n-a}{f^m-b}$ is reducible, or if $\overline{N}(r, f^n = a, f^m = b) \neq S(r, f)$, then there exists a small function α with respect to f such that $a = \alpha^n$ and $b = \alpha^m$, where $\overline{N}(r, f = a, g = b)$ denote the reduced counting function of all common zeros of $f - a$ and $g - b$.*

Proof. If $\frac{f^n-a}{f^m-b}$ is reducible, then there exists a small function α with respect to f such that $f^n - a = (f-\alpha)P(f)$ and $f^m = (f-\alpha)Q(f)$, where $P(f)$ and $Q(f)$ are polynomials in f of degree $n-1$ and $m-1$, respectively. Let

$$P(f) = c_{n-1}f^{n-1} + c_{n-2}f^{n-2} + \cdots + c_1 f + c_0,$$

where c_i $(i = 0, \ldots, n-1)$ are small functions of f. Then by comparing the coefficients of two side of the equation $f^n - a = (f-\alpha)P(f)$ yields

$$c_{n-1} = 1, \ c_j = \alpha^{n-1-j}, \ j = 0, 1, \ldots, n-2,$$

and $a = \alpha c_0$. Hence $a = \alpha^n$. Similarly, we can get $b = \alpha^m$.

Suppose that $\overline{N}(r, f^n = a, f^m = b) \neq S(r, f)$. Let z is a common zero of $f^n - a$ and $f^m - b$, i.e., $f^n(z) - a(z) = f^m(z) - b(z) = 0$. Thus z is a zero of $a^m - b^n$. Hence $a^m - b^n \equiv 0$. Since n, m are relatively prime, there exist integers s and t such that $sn + tm = 1$. Let $\alpha = a^s b^t$. Then we have $a = \alpha^n$ and $b = \alpha^m$. □

Now we prove Theorem 1.

From (1), we have $T(r, f) = T(r, g) + S(r, f)$. Let $S(r) = S(r, f) = S(r, g)$. Equation (1) can be rewritten as

$$(8) \qquad f_1 + f_2 = cb_2 - b_1,$$

where $f_1 = f^{n-m}(f^m + a_1)$, $f_2 = -cg^{n-m}(g^m + a_2)$. If $cb_2 - b_2 \not\equiv 0$, then by Nevanlinn's second fundamental theorem, we have $T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}(r, 1/f_1) + \overline{N}(r, 1/f_2) + S(r, f_1)$. Therefore,

$$nT(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^m + a_1}\right)$$
$$(9) \qquad + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^m + a_2}\right) + S(r).$$

It follows that

$$(10) \qquad nT(r, f) \leq (2m+3)T(r, f) + S(r),$$

which contradicts to $n > 2m + 3$. Hence $c = b_1/b_2$. And thus (1) becomes

$$(11) \qquad g^m(h^n - c) = -(a_1 h^{n-m} - ca_2),$$

where $h = f/g$.

If $h^n \not\equiv c$, then

$$(12) \qquad g^m = \frac{-(a_1 h^{n-m} - ca_2)}{h^n - c}.$$

If the rational function in h in the above equation is irreducible, then by Lemma 1 we have $\overline{N}(r, h^n = c, h^{n-m} = ca_2/a_1) = S(r)$. Therefore, "almost all" zeros of $h^n - c$ have multiplicities at least m. The above equation shows that $T(r, g) = \frac{n}{m}T(r, h) + S(r)$. Thus c is a small function of h. By Nevanlinns's second fundamental theorem, we get

$$nT(r,h) \leq \overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}\left(r, \frac{1}{h^n - c}\right) + S(r)$$
$$\leq 2T(r,h) + \frac{1}{m}N\left(r, \frac{1}{h^n - c}\right) + S(r)$$
$$\leq \left(2 + \frac{n}{m}\right) T(r,h) + S(r)$$

which leads to $n(m-1) \leq 2m$, a contradiction to the condition $n > 2m + 3$.

If the rational function in h in (12) is reducible, then there exists a small function α with respect to h such that $c = \alpha^n$ and $\frac{ca_2}{a_1} = \alpha^{n-m}$. Thus (12) can be written as

$$(13) \qquad g^m = -\frac{a_1}{\alpha^m} \frac{h_1^{n-m} - 1}{h_1^n - 1},$$

where $h_1 = h/\alpha$. Since n, m are relatively prime, equations $z^{n-m} - 1 = 0$ and $z^n - 1 = 0$ have different roots except for $z = 1$. Let r_j, $j = 1, \ldots, 2n - m - 2$ be all the roots of them. Then each of the r_j points of h_1 has a multiplicity at least m. Therefore, by the deficiency relation of h_1, we have

$$(14) \qquad \left(1 - \frac{1}{m}\right)(2n - m - 2) \leq 2,$$

i.e., $n \leq \frac{m^2+3m-2}{2(m-1)}$, which contradicts $n > 2m + 3$ when $m \geq 2$. Hence $h^n \equiv c$. It follows from (11) that $h^{n-m} = \frac{ca_2}{a_1}$. Thus $h^m = a_1/a_2$. This also completes the proof of Theorem 1.

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