Generalized Laplacians for generalized Poisson-Cauchy transforms on classical domains

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(Communicated by Shigefumi MORI, M.J.A., Nov. 13, 2006)

Abstract: We develop a group-theoretic method of generalizing the Laplace-Beltrami operators on the classical domains. In [18], we defined the generalized Poisson-Cauchy transforms on the classical domains. We show that the generalized Poisson-Cauchy transforms give us eigenfunctions of the generalized Laplacians defined in this paper.

Key words: Lie groups; analysis on homogeneous spaces; Lie group representations.

Introduction. In [11], Hua gave the explicit formulas of the Poisson kernel functions and the Cauchy kernel functions on classical domains. In the editor's foreword of the book [11], Graev says : "In carrying out his investigations by direct computation the author unfortunately does not make use of the possibilities of the group-theoretic aspect of the problems. Yet this group-theoretic aspect would have made possible a clearer understanding of many of the results, and would sometimes have simplified their proofs. Let $\mathfrak{R}$ be one of the domains considered in the book, and $\mathfrak{C}$ its characteristic manifold. Let z be a point in $\mathfrak{R}$ and C_z the group of those analytic automorphisms of $\mathfrak{R}$ which leave z invariant. It can be shown that the group C_z is transitive on $\mathfrak{C}$, i.e., transforms any point of $\mathfrak{C}$ into any other point. The measure on $\mathfrak{C}$ which is invariant under the transformations in C_z is then simply equal to the Poisson kernel".

About ten years ago this statement by Graev came to our notice. Then we started again the investigation of our "Poisson transforms" defined in [17] (see Section 2). In our previous paper [18], considering only classical domains and line bundles, we succeeded in computing explicitly the "Poisson transforms" and obtained the explicit formulas of the kernel functions which include, as special cases, not only Poisson kernel functions but also Cauchy kernel functions. In [18], we named these kernel functions the generalized Poisson-Cauchy kernel functions and "Poisson transforms" the generalized Poisson-Cauchy transforms.

In [11], Hua gave also the explicit formula of the Laplace-Beltrami operator for the Type I classical domain.

In this paper, we develop a group-theoretic method of obtaining a notion of "generalized Laplacians" which include, as a special case, the Laplace-Beltrami operator. And we show that the generalized Poisson-Cauchy transforms (defined in [18]) give us eigenfunctions of the generalized Laplacian.

In this paper, we follow the notation in [3, 6, 9, 16, 18, 23].

1. Harish-Chandra's realization of hermitian symmetric spaces as bounded domains. Let G/K be a hermitian symmetric space, where G is a non-compact semi-simple Lie group admitting a finite dimensional faithful representation and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively. Then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},\ \mathfrak{k} \cap \mathfrak{p} = \{0\},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. For any subset $\mathfrak{m}$ of $\mathfrak{g}_c$ we denote by $\mathfrak{m}_c$ the complex subspace of $\mathfrak{g}_c$ spanned by $\mathfrak{m}$. Since G/K is hermitian symmetric, there exist abelian subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ of $\mathfrak{g}_c$ such that

$$\mathfrak{p}_c = \mathfrak{p}_+ + \mathfrak{p}_-,\ \mathfrak{p}_+ \cap \mathfrak{p}_- = \{0\},\ \mathfrak{p}_+ = \overline{\mathfrak{p}}_-,$$
$$[\mathfrak{k}_c, \mathfrak{p}_+] \subset \mathfrak{p}_+,\ [\mathfrak{k}_c, \mathfrak{p}_-] \subset \mathfrak{p}_-.$$

Let G_c be the complexification of G with the Lie algebra $\mathfrak{g}_c$. We denote by K_c, P_+ and P_- the complex analytic subgroup of G_c corresponding to $\mathfrak{k}_c$, $\mathfrak{p}_+$ and

2000 Mathematics Subject Classiffcation. Primary 32A26, 22E46; Secondary 43A85, 32M15.

$\mathfrak{p}_-$, respectively. Then $P_+K_cP_-$ is an open subset of G_c and any point $w \in P_+K_cP_-$ is uniquely expressed as $w = p_+k_cp_-$ ($p_+ \in P_+,\ k_c \in K_c,\ p_- \in P_-$). This is called the Harish-Chandra decomposition. Put $U = K_cP_-$. Then U is a complex analytic subgroup of G_c and P_- a normal subgroup of U. Consider the complex homogeneous space G_c/U. Then G/K can be canonically identified with the open submanifold GU/U of G_c/U which is the G-orbit of the point U in G_c/U. Let B denote the Killing form of the Lie algebra $\mathfrak{g}$. Then B is positive definite on $\mathfrak{p}$. We introduce an inner product on the complex vector space $\mathfrak{p}_+$ by

$$(z_1, z_2) = B(z_1, -\overline{\theta(z_2)}) \quad (z_1, z_2 \in \mathfrak{p}_+),$$

where θ denotes the Cartan involution. One can show that $GU \subset P_+U$ and that there exists the unique bounded domain $\boldsymbol{D}$ in $\mathfrak{p}_+$ such that $GU = (\exp \boldsymbol{D})U$. For any w in GU, we denote by $z(w)$ and $u(w)$ the unique element of $\mathfrak{p}_+$ and U, respectively such that $w = (\exp z(w))u(w)$. For any $g \in G$ and $z \in \boldsymbol{D}$, we denote by $g[z]$ the unique element of $\boldsymbol{D}$ such that $g(\exp z)U = (\exp g[z])U$.

2. Homogeneous vector bundles and Poisson transforms. In this section, we would like to explain how we reached our concept of "Poisson transforms".

Let τ be a finite dimensional representation of K on a vector space V. Then τ is uniquely extended to a holomorphic representation of U which is trivial on P_-. We regard the complex Lie group G_c as the principal fiber bundle over the complex homogeneous space G_c/U with the structure group U. We denote by $\tilde{\boldsymbol{E}}_\tau$ the holomorphic vector bundle over G_c/U associated to τ. We denote by $\boldsymbol{E}_\tau$ the restriction of $\tilde{\boldsymbol{E}}_\tau$ to the open submanifold GU/U ($\cong G/K$) of G_c/U. Let $\boldsymbol{C}^{0,q}(\boldsymbol{E}_\tau)$ be the space of $\boldsymbol{C}^\infty$-differential forms of type $(0,q)$ with coefficients in $\boldsymbol{E}_\tau$. Then we have the $\overline{\partial}$-complex

$$\cdots \to \boldsymbol{C}^{0,q}(\boldsymbol{E}_\tau) \xrightarrow{\overline{\partial}} \boldsymbol{C}^{0,q+1}(\boldsymbol{E}_\tau) \to \cdots \quad .$$

Let $\boldsymbol{H}^{0,q}(\boldsymbol{E}_\tau)$ denote the space of harmonic forms in $\boldsymbol{C}^{0,q}(\boldsymbol{E}_\tau)$.

In [16], we obtained an anologue of the Borel-Weil-Bott theorem. In course of investigation, we found the following interesting facts. In a certain situation it happens that the Harish-Chandra's "Eisenstein integral" transforms K-finite sections of vector bundles on the "boundary" into harmonic forms on the bounded domain realized by Harish-Chandra. Here, the "boundary" means simply a homogeneous space G/P, where P is a parabolic subgroup of G. At this point, our purpose of the investigation was to find out a method of obtaining harmonic forms, namely eigenforms with zero-eigenvalue of the Laplacian $\tilde{\Box}^q$ for the $\overline{\partial}$-cohomology.

The problem was how to relate two spaces of $\boldsymbol{C}^\infty$-sections of different vector bundles.

Inspired by the Helgason's paper [7], we obtained an answer to this problem which is explained as follows (see [17]). Put $V_q = V \otimes \wedge^q \mathfrak{p}_-$. Then we have the representaion τ_q of K by putting $\tau_q(k) = \tau(k) \otimes \wedge^q Ad(k)|\mathfrak{p}_-$ ($k \in K$). Let ξ be a representation of P on V_q such that $\xi(m) = \tau_q(m)$ for all $m \in K \cap P$. Let $\boldsymbol{C}^\infty(G, V_q)$ denote the set of all V_q-valued $\boldsymbol{C}^\infty$-functions on G. We denote by $\boldsymbol{L}_\xi$ the vector bundle over G/P associated to ξ. Let $\boldsymbol{C}^\infty(G, V_q)_\xi$ be the set of all $\phi \in \boldsymbol{C}^\infty(G, V_q)$ such that

$$\phi(gp) = \xi(p)^{-1}\phi(g) \quad (g \in G,\ p \in P).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{L}_\xi$ is canonically identified with $\boldsymbol{C}^\infty(G, V_q)_\xi$. Let $\boldsymbol{C}^\infty(G, V_q)_\tau$ be the set of all $f \in \boldsymbol{C}^\infty(G, V_q)$ such that

$$f(gk) = \tau_q(k)^{-1}f(g) \quad (g \in G,\ k \in K).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{C}^{0,q}(\boldsymbol{E}_\tau)$ is canonically identified with $\boldsymbol{C}^\infty(G, V_q)_\tau$.

Thus we realized two spaces of $\boldsymbol{C}^\infty$-sections of different vector bundles $\boldsymbol{L}_\xi$ and $\boldsymbol{E}_\tau \otimes \wedge^q \overline{T^*(\boldsymbol{D})}$ as subspaces of the same vector space $\boldsymbol{C}^\infty(G, V_q)$.

Remark that G acts on both spaces $\boldsymbol{C}^\infty(G, V_q)_\xi$ and $\boldsymbol{C}^\infty(G, V_q)_\tau$ by the induced action of the left translation of g^{-1} ($g \in G$). Thus we obtain two representations of G on spaces $\boldsymbol{C}^\infty(G, V_q)_\xi$ and $\boldsymbol{C}^\infty(G, V_q)_\tau$.

Now the easiest way to get an intertwining operator is given by considering an integral operator

$$\boldsymbol{P}_{\tau_q,\xi} \ :\ \boldsymbol{C}^\infty(G, V_q)_\xi \ \ni \phi \longmapsto f \in \boldsymbol{C}^\infty(G, V_q)_\tau$$

defined by

$$f(g) = \int_K \tau_q(k)\phi(gk)dk \quad (g \in G).$$

In [17], we called this the "Poisson transform". Computing explicitly $\boldsymbol{P}_{\tau_q,\xi}$, we found many examples where $\boldsymbol{P}_{\tau_q,\xi}$ gives us not only harmonic forms but also eigenforms (with non-zero eigenvalues in general) of the Laplacian $\tilde{\Box}^q$. Since then we have occasionally tried to find out suitable conditions that

$\boldsymbol{P}_{\tau_q,\xi}$ produces eigenforms of $\tilde{\Box}^q$ (cf. examples in [5, 17]), but could not get any satisfactory result.

Here we like to remark that the "Poisson transform" can be defined in the most general way as follows. Starting with an arbitrary symmetric space G/K and a parabolic subgroup P of G, we can define the "Poisson transform" by the above integral operator, replacing τ_q with an arbitrary finite dimensional representation of K. We remind the reader that the Euclidean space is one of the simplest symmetric spaces. We dealt with the case of the Euclidean space in [4, 15] and an affine symmetric space in [10].

3. Realizations of a classical domain and the Shilov boundary as various homogeneous spaces. From now on, we focus our attention to the case that the symmetric space is a classical domain and that the representation of K is of one dimension.

There are four types of classical domains. Let $\boldsymbol{D}$ be one of such domains. Then $\boldsymbol{D}$ is a homogeneous space of a simple Lie group G which admits a finite dimensional faithful representation. Let K be the isotropy subgroup of G at the origin. Then K is a maximal compact subgroup of G.

The key point of our group-theoretic method is to redefine $\boldsymbol{D}$ by realizing G/K as a bounded domain (which we denote by the same notation $\boldsymbol{D}$) in $\mathfrak{p}_+$ realized by Harich-Chandra which we described in Section 1.

G acts on $G/K \cong GU/U \cong \boldsymbol{D}$ by the following commutative diagram.

$$\begin{array}{ccccc} G/K & \cong & GU/U & \cong & \boldsymbol{D} \\ \Cup & & \Cup & & \Cup \\ gK & \longmapsto & gU & \longmapsto & g[0]=z \\ \downarrow & & \downarrow & & \downarrow \\ g_1g/K & \longmapsto & g_1g/U & \longmapsto & g_1g[0]=g_1[z] \\ \Cap & & \Cap & & \Cap \\ G/K & \cong & GU/U & \cong & \boldsymbol{D}, \end{array}$$

g_1 being any element of G.

We fix a point $\mu U \in G_c/U$ such that μU belongs to the boundary of GU/U and that the G-orbit of μU is compact. Then the isotropy subgroup of G at the point μU of G_c/U is a maximal parabolic subgroup of G which we denote by P. Put $u_0 = z(\mu)$ and $\mu_0 = \exp u_0$. Then clearly we get $\mu U = \mu_0 U = (\exp u_0)U$ which implies that $G \cap \mu U \mu^{-1} = G \cap \mu_0 U \mu_0^{-1} = P$. Put

$$\check{\boldsymbol{S}} = \{u \in \mathfrak{p}_+ \,;\, (\exp u)U \in G\mu_0 U/U\}\,.$$

Then $\check{\boldsymbol{S}}$ is the Shilov boundary of $\boldsymbol{D}$. For any $g \in G$ and $u \in \check{\boldsymbol{S}}$, we denote by $g[u]$ the unique element of $\check{\boldsymbol{S}}$ such that $g(\exp u)U = (\exp g[u])U$. G acts on $G/P \cong G\mu_0 U/U \cong \check{\boldsymbol{S}}$ by the following commutative diagram.

$$\begin{array}{ccccc} G/P & \cong & G\mu_0U/U & \cong & \check{\boldsymbol{S}} \\ \Cup & & \Cup & & \Cup \\ g/P & \longmapsto & g\mu_0/U & \longmapsto & g[u_0]=u \\ \downarrow & & \downarrow & & \downarrow \\ g_1g/P & \longmapsto & g_1g\mu_0/U & \longmapsto & g_1g[u_0]=g_1[u] \\ \Cap & & \Cap & & \Cap \\ G/P & \cong & G\mu_0U/U & \cong & \check{\boldsymbol{S}}, \end{array}$$

g_1 being any element of G.

4. Homogeneous line bundles and onto-isomorphisms between spaces of $\boldsymbol{C}^\infty$-sections. Let τ be a character of K. We define $\tilde{\boldsymbol{E}}_\tau$ and $\boldsymbol{E}_\tau$ in the same way as in Section 2. Let $\boldsymbol{C}^\infty(\tilde{\boldsymbol{E}}_\tau)$ be the set of all $h \in \boldsymbol{C}^\infty(G_c)$ such that

$$h(wu) = \tau(u)^{-1}h(w) \quad (w \in G_c,\ u \in U).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\tilde{\boldsymbol{E}}_\tau$ is identified with $\boldsymbol{C}^\infty(\tilde{\boldsymbol{E}}_\tau)$. Let $\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$ be the set of all $h \in \boldsymbol{C}^\infty(GU)$ such that

$$h(wu) = \tau(u)^{-1}h(w) \quad (w \in GU,\ u \in U).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{E}_\tau$ is identified with $\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$. Let η be a $\boldsymbol{C}^\infty$-character of U such that the restriction of η to K coincides with τ. We denote by $\tilde{\boldsymbol{L}}_\eta$ the $\boldsymbol{C}^\infty$-line bundle on G_c/U associated to η. Let $\boldsymbol{C}^\infty(\tilde{\boldsymbol{L}}_\eta)$ be the set of all $\psi \in \boldsymbol{C}^\infty(G_c)$ such that

$$\psi(wu) = \eta(u)^{-1}\psi(w) \quad (w \in G_c,\ u \in U).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\tilde{\boldsymbol{L}}_\eta$ is identified with $\boldsymbol{C}^\infty(\tilde{\boldsymbol{L}}_\eta)$. We denote by $\boldsymbol{L}_\eta$ the restriction of $\tilde{\boldsymbol{L}}_\eta$ to the compact submanifold $G\mu_0 U/U$ of G_c/U. Let $\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)$ be the set of all $\psi \in \boldsymbol{C}^\infty(G\mu_0 U)$ such that

$$\psi(wu) = \eta(u)^{-1}\psi(w) \quad (w \in G\mu_0 U,\ u \in U).$$

Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{L}_\eta$ is identified with $\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)$. We define a $\boldsymbol{C}^\infty$-character ξ of P by

$$\xi(p) = \eta(\mu_0^{-1}p\mu_0) \quad (p \in P).$$

Let $\boldsymbol{C}^\infty(G)_\tau$ and $\boldsymbol{C}^\infty(G)_\xi$ be the set of all $f \in \boldsymbol{C}^\infty(G)$ and $\phi \in \boldsymbol{C}^\infty(G)$ such that

$$f(gk) = \tau(k)^{-1}f(g) \quad (g \in G,\ k \in K)$$

and

$$\phi(gp) = \xi(p)^{-1}\phi(g) \quad (g \in G,\ p \in P),$$

respectively. Then we obtain the following four onto-isomorphisms.

$$\boldsymbol{C}^\infty(\boldsymbol{E}_\tau) \ni h \longmapsto f \in \boldsymbol{C}^\infty(G)_\tau,\ f(g) = h(g),$$
$$\boldsymbol{C}^\infty(\boldsymbol{E}_\tau) \ni h \longmapsto F \in \boldsymbol{C}^\infty(\boldsymbol{D}),\ F(z) = h(\exp z),$$
$$\boldsymbol{C}^\infty(\boldsymbol{L}_\eta) \ni \psi \longmapsto \phi \in \boldsymbol{C}^\infty(G)_\xi,\ \phi(g) = \psi(g\mu_0),$$
$$\boldsymbol{C}^\infty(\boldsymbol{L}_\eta) \ni \psi \longmapsto \varPhi \in \boldsymbol{C}^\infty(\check{\boldsymbol{S}}),\ \varPhi(u) = \psi(\exp u),$$

where $g \in G$, $z \in \boldsymbol{D}$ and $u \in \check{\boldsymbol{S}}$.

5. Generalized Poisson-Cauchy transforms and generalized Laplacians. In [18], we defined the generalized Poisson-Cauchy transform :

$$\boldsymbol{P}_{\tau,\xi}\ :\ \boldsymbol{C}^\infty(G)_\xi \ni \phi \longmapsto f \in \boldsymbol{C}^\infty(G)_\tau$$

by

$$f(g) = \int_K \tau(k)\phi(gk)dk \quad (g \in G),$$

where dk is the normalized Haar measure of K.

We define $\boldsymbol{P}_{\tau,\eta}$ in such a way that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{L}_\eta) & \cong & \boldsymbol{C}^\infty(G)_\xi & \\ \boldsymbol{P}_{\tau,\eta} & \downarrow & & \downarrow & \boldsymbol{P}_{\tau,\xi} \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(G)_\tau. & \end{array}$$

We call $\boldsymbol{P}_{\tau,\eta}$ the generalized Poisson-Cauchy transform (with respect to the pair $(\boldsymbol{L}_\eta,\ \boldsymbol{E}_\tau)$). For any $g \in G$, we define

$$\pi_\tau(g)\ :\ \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) \ni h \longmapsto \pi_\tau(g)h \in \boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$$

by

$$(\pi_\tau(g)h)(w) = h(g^{-1}w) \quad (w \in GU).$$

For any $g \in G$, we define

$$\pi_\eta(g)\ :\ \boldsymbol{C}^\infty(\boldsymbol{L}_\eta) \ni \psi \longmapsto \pi_\eta(g)\psi \in \boldsymbol{C}^\infty(\boldsymbol{L}_\eta)$$

by

$$(\pi_\eta(g)\psi)(w) = \psi(g^{-1}w) \quad (w \in G\mu_0 U).$$

Then π_τ and π_η are representations of G on $\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$ and $\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)$, respectively. Let $d\pi_\tau$ and $d\pi_\eta$ be the differential of π_τ and π_η, respectively. Then $d\pi_\tau$ and $d\pi_\eta$ are representations of $\mathfrak{g}$ which are canonically extended to the representations of the universal enveloping algebra of $\mathfrak{g}_c$.

We denote by Ω the Casimir operator (see [9, 23]). Then $d\pi_\tau(\Omega)$ is an invariant differential operator on $\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$ which we denote by $\Delta_{\boldsymbol{E}_\tau}$. We call $\Delta_{\boldsymbol{E}_\tau}$ the generalized Laplacian (on $\boldsymbol{E}_\tau$).

We can prove that $\boldsymbol{P}_{\tau,\eta}$ is an intertwining operator between the representations π_τ and π_η.

Proposition 1. *For any $g \in G$, the following diagram is commutative.*

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{L}_\eta) & \xrightarrow{\boldsymbol{P}_{\tau,\eta}} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \\ \pi_\eta(g) & \downarrow & & \downarrow & \pi_\tau(g) \\ & \boldsymbol{C}^\infty(\boldsymbol{L}_\eta) & \xrightarrow{\boldsymbol{P}_{\tau,\eta}} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau). & \end{array}$$

From this commutative diagram we obtain the following corollary.

Corollary 1.

$$\boldsymbol{P}_{\tau,\eta} \circ d\pi_\eta(\Omega) = d\pi_\tau(\Omega) \circ \boldsymbol{P}_{\tau,\eta}.$$

The crucial point of our method is due to the fact that $d\pi_\eta(\Omega)$ turns out to be a scalar operator.

Proposition 2. *Let $\boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)}$ be the identity operator on $\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)$. Then there exists the unique complex number c_η such that*

$$d\pi_\eta(\Omega) = c_\eta \boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)}.$$

We put

$$\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)_{c_\eta} = \left\{h \in \boldsymbol{C}^\infty(\boldsymbol{E}_\tau);\ \Delta_{\boldsymbol{E}_\tau} h = c_\eta h\right\}.$$

The next corollary follows immediately from Corollary 1 and Proposition 2 and means that $\boldsymbol{P}_{\tau,\eta}$ gives us eigenfunctions (eigensections more precisely) of $\Delta_{\boldsymbol{E}_\tau}$.

Corollary 2.

$$\boldsymbol{P}_{\tau,\eta}(\boldsymbol{C}^\infty(\boldsymbol{L}_\eta)) \ \subset\ \boldsymbol{C}^\infty(\boldsymbol{E}_\tau)_{c_\eta}.$$

6. Deduction of Theorem. We normalize the K-invariant measure on $\check{\boldsymbol{S}}$ such that for any $\boldsymbol{C}^\infty$-function $\varPhi$ on $\check{\boldsymbol{S}}$ we have

$$\int_K \varPhi(k[u_0])dk = \int_{\check{\boldsymbol{S}}} \varPhi(u)du,$$

where dk is the normalized Haar measure of K.

We define $\mathcal{P}_{\tau,\eta}$ in such a way that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{L}_\eta) & \cong & \boldsymbol{C}^\infty(\check{\boldsymbol{S}}) & \\ \boldsymbol{P}_{\tau,\eta} & \downarrow & & \downarrow & \mathcal{P}_{\tau,\eta} \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}). & \end{array}$$

We call $\mathcal{P}_{\tau,\eta}$ the generalized Poisson-Cauchy transform (with respect to the pair $(\check{\boldsymbol{S}},\ \boldsymbol{D})$).

In [18], we proved the following lemma.

Lemma 1. *For any $\varPhi \in \boldsymbol{C}^\infty(\check{\boldsymbol{S}})$, we have*

$$(\mathcal{P}_{\tau,\eta}(\varPhi))(z) \ =\ \int_{\check{\boldsymbol{S}}} \mathcal{K}_{\tau,\eta}(z,u)\varPhi(u)du \quad (z \in \boldsymbol{D}),$$

where $\mathcal{K}_{\tau,\eta}(z,u)$ is the generalized Poisson-Cauchy kernel function defined in [18].

(In our proof of this lemma we use our assumption that both τ and η are characters so that for any $u_1, u_2 \in U$ $\tau(u_1)$ and $\eta(u_2)$ are commutative.)

We define Δ_τ in such a way that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}) & \\ \Delta_{\boldsymbol{E}_\tau} & \downarrow & & \downarrow & \Delta_\tau \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}). & \end{array}$$

We call Δ_τ the generalized Laplacian (on $\boldsymbol{D}$).

Since $\boldsymbol{D} \cong G/K$, the tangent space of $\boldsymbol{D}$ at the origin is canonically identified with $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. Notice that the Killing form B is positive definite on $\mathfrak{p}$ and that B is invariant by the adjoint action of G. Then it is easy to see that B defines a G-invariant riemannian metric on $\boldsymbol{D}$. We denote by Δ the Laplace-Beltrami operator defined by this riemannian metric. Then we can prove the following proposition.

Proposition 3. *In case the line bundle $\boldsymbol{E}_\tau$ is trivial, Δ_τ coincides with the Laplace-Beltrami operator Δ.*

Using the propositions and the corollaries in the previous section, from Lemma 1 we can now deduce the following theorem.

Theorem. *For any $\Phi \in \boldsymbol{C}^\infty(\check{\boldsymbol{S}})$, define*

$$F(z) = \int_{\check{\boldsymbol{S}}} \mathcal{K}_{\tau,\eta}(z,u)\Phi(u)du \quad (z \in \boldsymbol{D}).$$

Then we have

$$\Delta_\tau F(z) = c_\eta F(z) \quad (z \in \boldsymbol{D}),$$

where c_η is the complex number given in Proposition 2.

This theorem asserts that for any fixed $u \in \check{\boldsymbol{S}}$ the generalized Poisson-Cauchy kernel function $\mathcal{K}_{\tau,\eta}(z,u)$ is an eigenfunction of the generalized Laplacian Δ_τ.

7. Explicit formulas of generalized Laplacians and eigenvalues. In [18], we computed the explicit formulas of the generalized Poisson-Cauchy kernel functions for each type of classical domains. For example, the generalized Poisson-Cauchy kernel function (associated with the characters τ_ℓ and $\eta_{\ell,s}$) for Type **I** is given explicitly by

$$\mathcal{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{1}{\det(I_m - u^*z)^\ell}\left(\frac{\det(I_m - z^*z)}{|\det(I_m - u^*z)|^2}\right)^{n-(\ell+s)/2}.$$

We can compute the generalized Laplacian explicitly (up to a constant factor and scalar operators)

$$\Delta_{\tau_\ell} = \mathrm{Tr}(\det(I_m - z^*z)^{-\ell}(I_m - z^*z)\partial_z \det(I_m - z^*z)^\ell \cdot (I_n - zz^*) \cdot (\partial_z)^*),$$

where the function between two dots "$\cdot$" should not be differentiated. We can compute explicitly the eigenvalue $c_{\eta_{\ell,s}}$.

$$\Delta_{\tau_\ell}\mathcal{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{m(s+\ell-2n)(s-\ell)}{4}\mathcal{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) \quad (u \in \check{\boldsymbol{S}}).$$

We will compute explicitly the eigenvalues c_η for each type of the classical domains in the forthcoming paper [13].

8. Remarks on further development of the results. We like to mention the possibility of the further development of our results.

In [2], Graev gave very interesting results where he computed explicitly the Laplace-Beltrami operator on the outside of the Type I bounded domain of complex dimension 2 (which is an affine symmetric space). One can construct the explicit form of the analogue of the Poisson-Cauchy kernel function.

In [14], we considered the Martin boundary (where the isotropy subgroup is minimal parabolic) and all invariant differential operators on the symmetric space. There are still a great deal of works to be done in the more general cases of boundaries, symmetric spaces and invariant differential operators (cf. [1, 8, 10, 12, 20–22]).

In [19], we considered vector bundles instead of line bundles. For a special choice of a vector bundle on the classical domain of Type I ($m = n = 2$), we computed explicitly a matrix valued "Poisson-Cauchy kernel function" $\mathcal{K}_{\tau_\sigma,\eta_s}(z,u)$ and a matrix valued invariant differential operator $\hat{\Delta}_z$.

$$\mathcal{K}_{\tau_\sigma,\eta_s}(z,u) = \left(\frac{\det(I_2 - z^*z)}{|\det(I_2 - u^*z)|^2}\right)^{2-s/2}\left(\frac{\det(I_2 - z^*z)}{\det(I_2 - u^*z)}\right)^{-2} \times (I_2 - z^*z)(I_2 - u^*z)^{-1},$$

$$\hat{\Delta}_z = (I_2 - z^*z)\partial_z \cdot (I_2 - zz^*) \cdot (\partial_z)^*.$$

Moreover we showed that

$$\hat{\Delta}_z\mathcal{K}_{\tau_\sigma,\eta_s}(z,u) = \frac{s(s-4)}{4}\mathcal{K}_{\tau_\sigma,\eta_s}(z,u) \quad (u \in \check{\boldsymbol{S}}).$$

We will discuss the general case of this example elsewhere.

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