Cofree embeddings of algebraic tori preserving canonical sheaves

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Abstract: Let $\varrho : G \to GL(V)$ be a finite dimensional rational representation of a diagonalizable algebraic group G over an algebraically closed field K of characteristic zero. Using a minimal paralleled linear hull (W, w) of ϱ defined in [N4], we show the existence of a cofree representation $\widetilde{G_w} \hookrightarrow GL(W)$ such that $\varrho(G_w) \subseteq \widetilde{G_w}$ and $W//G_w \to W//\widetilde{G_w}$ is divisorially unramified is equivalent to the Gorensteinness of $V//G$.

Key words: Cofree representations; algebraic tori; character groups; canonical modules; Gorenstein rings.

1. Introduction. Without specifying, G will always stand for a reductive affine algebraic group whose identity component is an *algebraic torus* over an algebraically closed field K of characteristic zero. Let $\mathfrak{X}(G)$ stand for the rational character group of G over K which is regarded as an additive group. For an affine variety X over K, $\mathcal{O}(X)$ denotes the K-algebra of all regular functions on X. When a regular action of G on an affine variety X (abbr. (X, G)) is given, we denote by $X//G$ the algebraic quotient of X under the action of G and by $\pi_{X,G}$ the quotient map $X \to X//G$. For $\psi \in \mathfrak{X}(G)$, let $\mathcal{O}(X)_\psi$ be the set $\{f \in \mathcal{O}(X) \mid \sigma(f) = \psi(\sigma) \cdot f \ \ (\forall \sigma \in G)\}$, which is regarded as an $\mathcal{O}(X)^G$-module. A regular action (X, G) is said to be *stable*, if X contains a nonempty open subset consisting of closed G-orbits. Let $X_{\rm st}$ denote the affine variety defined by $\mathcal{O}(X_{\rm st}) = \mathcal{O}(X)_{\rm st}$, where $\mathcal{O}(X)_{\rm st}$ is the K-subalgebra of $\mathcal{O}(X)$ generated by $\mathcal{O}(X)_\chi$'s such that $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{-\chi} \neq \{0\}$, $\chi \in \mathfrak{X}(G^0)$ (cf. [N1]). Then the induced action $(X_{\rm st}, G)$ is stable, for any (X, G). Consider a finite dimensional rational G-module V. A pair (W, w) is defined to be a *paralleled linear hull* of (V, G), if W is a G-submodule of $V_{\rm st}$ such that G is diagonalizable on the quotient module $V_{\rm st}/W$, w is a nonzero vector of $V_{\rm st}$ satisfying the condition $W \cap < G \cdot w >_K = \{0\}$ and the G_w-equivariant morphism

$$(\bullet + w) : W \ni x \mapsto x + w \in V_{\rm st}$$

induces the isomorphism

$$\pi_{V_{\rm st}//G_w, V//G} \circ (\bullet + w)//G_w : W//G_w \xrightarrow{\sim} V_{\rm st}//G.$$

Here $(\bullet + w)//G_w : W//G_w \longrightarrow V_{\rm st}//G_w$ is the quotient of $(\bullet + w)$ modulo G_w and $\pi_{V_{\rm st}//G_w, V//G}$: $V_{\rm st}//G_w \longrightarrow V_{\rm st}//G$ is associated with the inclusion $\mathcal{O}(V_{\rm st})^G \hookrightarrow \mathcal{O}(V_{\rm st})^{G_w}$. A paralleled linear hull (W_0, w_o) of (V, G) is said to be *minimal*, if W_0 is minimal with respect to inclusions in the set consisting of all subspaces W's of $V_{\rm st}$ such that (W, w)'s are paralleled linear hulls of (V, G) for some w 's. An element $\sigma \in G$ is said to be a *pseudo-reflection* on V, if $\dim(\sigma - 1)(V^\vee) = {\rm ht}((\sigma - 1)(V^\vee) \cdot \mathcal{O}(V) \cap \mathcal{O}(V)^G) \leq 1$, where $V^\vee$ is the dual space of V over K. We have the following fundamental result for minimal paralleled hulls:

Theorem 1.1 (cf. [N4]). *Suppose that G equals to the centralizer $Z_G(G^0)$ of G^0 in G. Let (W, w) be a minimal paralleled linear hull of (V, G). Then*

(1) *π_{W,G_w} is no-blowing-up of codimension one (for definition, cf. [F]) and G_w acts transitively on the set of irreducible components of codimension one in W of the fibre of each irreducible closed subvariety of $W//G_w$ of codimension one under the morphism π_{W,G_w}.*

(2) *$\mathfrak{X}(G_w/\mathcal{R}_W(G_w)) \cong {\rm Cl}(\mathcal{O}(V)^G)$, where $\mathcal{R}_W(G_w)$ denotes the subgroup of G_w generated by all pseudo-reflections of G_w on W.*

Conversely, the conclusion (1) of (1.1) characterizes minimality of the paralleled linear full (W, G_w) of (V, G) under the condition $Z_G(G^0) = G$. For a *diagonalizable* G, the following results are obtained. In Sect. 3, we will show the existence of a cofree representation $\widetilde{G_w} \hookrightarrow GL(W)$ such that $G_w|_W \subseteq \widetilde{G_w}$ and $W//G_w \to W//\widetilde{G_w}$ is divisorially unramified is

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equivalent to the fact that $V//G$ is Gorenstein. Some examples of the main result (2.6) can be found in Sect. 2.

The symbol $\sharp(\ \circ\ )$ stands for the cardinality of the set $\circ$ and let $\mathbf{Z}_0$ denote the additive monoid of non-negative integers. For a mapping $\varphi : A \to B$ and a subset $A' \subseteq A$, let $\varphi|_{A'}$ denote the restriction of φ to A' and, for a set Ω of mappings $\varphi : A \to B$, let $\Omega|_{A'}$ be the set of restrictions $\varphi|_{A'}$'s $(\varphi \in \Omega)$.

2. Cofree embeddings. If Λ is a subset of $\mathfrak{X}(G)$, let $\mathbf{Z}_0 \cdot \Lambda$ (resp. $\mathbf{Z}_+ \cdot \Lambda$) denote the set of all linear combinations of any finite subset of Λ with coefficients in $\mathbf{Z}_0$ (resp. $\mathbf{Z}_+$) in $\mathfrak{X}(G)$, where $\mathbf{Z}_+ = \mathbf{N}$ and $\mathbf{Z}_0 \cdot \emptyset$ means $\{0\}$. For any $\chi \in \mathfrak{X}(G)$ and a rational G-module V, let $V_\chi = \{x \in V \mid \sigma(x) = \chi(\sigma) \cdot x \ \ (\forall \sigma \in G)\}$ denote the subspace of χ-*invariants* or *relative invariants* of G with respect to χ in V. For a rational G-module V, let $V^\vee$ be the dual module on which G acts naturally and $\mathcal{W}(V, G)$ denote the set of all weights of G on V (i.e., $\{\chi \in \mathfrak{X}(G) \mid V_\chi \neq \{0\}\}$).

Definition 2.1. A subset Λ of $\mathcal{W}(V_{\mathrm{st}}, G)$ of a finite dimensional rational G-module V is said to be G-*removable* on V_{st}, if $\dim(V_{\mathrm{st}})_{\chi|_{G^0}} = 1$ $(\forall \chi \in \Lambda)$ and

$$(\mathbf{Z}_+ \cdot \Lambda'|_{G^0}) \bigcap (\mathbf{Z}_0 \cdot (\mathcal{W}(V_{\mathrm{st}}, G^0) \backslash \Lambda'|_{G^0})) = \emptyset$$

for any non-empty subset Λ' of Λ. Clearly any G-removable subset does not contains the trivial character 0. We say Λ is a maximal G-removable subset of $\mathcal{W}(V_{\mathrm{st}}, G)$, if it is maximal with respect to inclusions in the set of all G-removable subsets of $\mathcal{W}(V_{\mathrm{st}}, G)$.

Let Λ be a subset of $\mathcal{W}(V_{\mathrm{st}}, G)$ and y_χ a nonzero element in $((V_{\mathrm{st}})^\vee)_{-\chi}$ for each $\chi \in \Lambda$. The set $\{y_\chi \mid \chi \in \Lambda\}$ is said to be $(\mathcal{O}(V_{\mathrm{st}}), G)$-free, if, for any $a_\chi \in \mathbf{Z}_0$, there exists a rational character $\psi \in \mathfrak{X}(G)$ such that

$$\mathcal{O}(V_{\mathrm{st}})_\psi = \mathcal{O}(V_{\mathrm{st}})^G \cdot \prod_{\chi \in \Lambda} y_\chi^{a_\chi}.$$

Then we have

Proposition 2.2. *For any subset* Λ *of* $\mathcal{W}(V_{\mathrm{st}}, G)$*, it is* G*-removable on* V_{st} *if and only if* $\{y_\chi \mid \chi \in \Lambda\}$ *is* $(\mathcal{O}(V_{\mathrm{st}}), G)$*-free.*

Proof. We easily see that the set $\{y_\chi \mid \chi \in \Lambda\}$ is $(\mathcal{O}(V_{\mathrm{st}}), G)$-free if and only if it is $(\mathcal{O}(V_{\mathrm{st}}), G^0)$-free (cf. [N4]). Suppose that Λ is G-removable in $\mathcal{W}(V_{\mathrm{st}}, G)$. As

$$((V_{\mathrm{st}})^\vee)_{-\chi|_{G^0}} = K \cdot y_\chi = ((V_{\mathrm{st}})^\vee)_{-\chi},$$

we see that $\chi|_{G^O}$, $\chi \in \Lambda$, are different each other. Let us denote by $\{z_1, \ldots, z_l\}$ a K-basis of the K-subspace $\sum_{\psi \in \mathcal{W}(V_{\mathrm{st}}, G^0) \backslash \Lambda|_{G^0}} ((V_{\mathrm{st}})^\vee)_{-\psi}$ of $(V_{\mathrm{st}})^\vee$ consisting of relative invariants of G^0. Let a_χ, $\chi \in \Lambda$, be any nonnegative integers. For $b_\chi \in \mathbf{Z}_0$ $(\chi \in \Lambda)$ and $c_i \in \mathbf{Z}_0$ $(1 \leq i \leq l)$, the condition that

$$\prod_{\chi \in \Lambda} y_\chi^{b_\chi} \cdot \prod_{i=1}^{l} z_i^{c_i} \in \mathcal{O}(V_{\mathrm{st}})_{\sum_{\chi \in \Lambda} a_\chi \cdot \chi|_{G^0}}$$

is equivalent to

$$\sum_{\chi \in \Lambda, b_\chi \geq a_\chi} (b_\chi - a_\chi) \cdot \chi|_{G^0} + \sum_{i=1}^{l} c_i \cdot \psi_i = \sum_{\chi \in \Lambda, a_\chi > b_\chi} (a_\chi - b_\chi) \cdot \chi|_{G^0},$$

where $\psi_i \in \mathcal{W}(V_{\mathrm{st}}, G^0)$ such that $z_i \in ((V_{\mathrm{st}})^\vee)_{-\psi_i}$. Thus (2.1) implies that $b_\chi \geq a_\chi$ for all $\chi \in \Lambda$, which implies

$$\prod_{\chi \in \Lambda} y_\chi^{b_\chi} \cdot \prod_{i=1}^{l} z_i^{c_i} \in \mathcal{O}(V_{\mathrm{st}})^{G^0} \cdot \prod_{\chi \in \Lambda} y_\chi^{a_\chi}$$

and the equality

$$\mathcal{O}(V_{\mathrm{st}})_{\sum_{\chi \in \Lambda} a_\chi \cdot \chi|_{G^0}} = \mathcal{O}(V_{\mathrm{st}})^{G^0} \cdot \prod_{\chi \in \Lambda} y_\chi^{a_\chi}.$$

The proof of "*if part*" of the assertion in Proposition 2.2 is left to the reader. □

For a subset Ω of V_{st} or $(V_{\mathrm{st}})^\vee$, let $\Omega^\perp$ be the set of all elements orthogonal to Ω under the canonical pairing

$$V_{\mathrm{st}} \times (V_{\mathrm{st}})^\vee \to K.$$

Combining (2.2) with [N4], we immediately have

Corollary 2.3. *For a* G*-submodule* W *of* V_{st} *such that* G *is diagonalizable on* V/W*, there exists a vector* $w \in V_{\mathrm{st}}$ *satisfying that* (W, w) *is a paralleled linear hull if and only if*

$$W = \left(\sum_{\chi \in \Lambda} ((V_{\mathrm{st}})^\vee)_{-\chi} \right)^\perp$$

for a G*-removable subset* Λ *of* $\mathcal{W}(V_{\mathrm{st}}, G)$ *on* V_{st}*. Furthermore, in this notation,* (W, w) *is a minimal paralleled linear hull of* (V, G) *if and only if* Λ *is a maximal* G*-removable subset of* $\mathcal{W}(V_{\mathrm{st}}, G)$ *on* V_{st}*.* □

For a Cohen-Macaulay $\mathbf{Z}_0$-graded domain R defined over K, the graded canonical module of R is denoted by ω_R.

Theorem 2.4 (G. Kempf - R. P. Stanley - V. I. Danilov (e.g., [D, S2, TE])). *Let $\varrho : D \to GL(V)$ be a finite dimensional stable rational representation of a diagonalizable group D. Then the canonical module $\omega_{\mathcal{O}(V)^D}$ of the $\mathbf{Z}_0$-graded Cohen-Macaulay algebra $\mathcal{O}(V)^D$ is isomorphic to the graded module $\mathcal{O}(V)_{(\det_V)|_D}(-\dim V)$ of invariants relative to $\det_V|_D$ in $\mathcal{O}(V)$.*

Definition 2.5. For a finite dimensional rational representation $\phi : H \to GL(W)$ of a diagonalizable group H, a faithful rational representation $\widetilde{\phi} : \widetilde{H} \to GL(W)$ of a diagonalizable group $\widetilde{H}$ is defined to be a *cofree embedding of* $\phi : H \to GL(W)$ (or *of* (W, H)), if the following conditions are satisfied:

(1) $\phi(H)$ $(= H|_W) \subseteq \widetilde{\phi}(\widetilde{H})$ and $\phi(\mathcal{R}_W(H)) = \widetilde{\phi}(\mathcal{R}_W(\widetilde{H}))$.

(2) The representation $\widetilde{\phi}$ is stable and cofree.

A cofree embedding $\widetilde{\phi} : \widetilde{H} \to GL(W)$ of $\phi : H \to GL(W)$ is said to be *canonical*, if $\widetilde{\phi}(\widetilde{H})$ is minimal in $\psi(L)$'s for all cofree embeddings $\psi : L \to GL(W)$ of ϕ.

Theorem 2.6. *Suppose that G is a diagonalizable group and (V, G) is a finite dimensional representation of G. Then the following conditions are equivalent:*

(1) *$V//G$ is a Gorenstein variety.*

(2) *For a minimal paralleled linear hull (W, w) of (V, G), there exists a canonical cofree embedding $(W, \widetilde{H})$ of (W, G_w).*

If these conditions hold, then

$$\omega_{\mathcal{O}(W)^{\widetilde{H}}} \cdot \mathcal{O}(W)^{G_w} = \omega_{\mathcal{O}(W)^{G_w}}$$

and $\mathcal{O}(W)^{\widetilde{H}}$ is generated by a part of a minimal generating system of $\mathcal{O}(W)^{G_w}$ consisting of monomials of a K-basis of $W^\vee$ on which $\widetilde{H}$ is represented as a diagonal group.

2.7. Examples for (2.6). In order to explain the content of this theorem, we now give the following examples, in which $\{Y_1, \dots, Y_n\}$ denotes a K-basis of $V^\vee$ such that G is diagonal on this basis.

Example 2.7.1. For any $N = \prod Y_i^{c_i} \in \mathcal{O}(V)$, we denote $\mathrm{supp}_{\{Y_i\}}(N)$ by the set $\{Y_i \mid c_i > 0\}$. Suppose that $\mathcal{O}(V)^G$ has a homogeneous system $\{N_1, \dots, N_d\}$ $(d \geq 1)$ of parameters consisting of monomials of that basis (if $\dim \mathcal{O}(V)^G \leq 2$, this condition always holds (e.g., [TE])). Then we must have $V_{\mathrm{st}} = (\sum_{Y_k \in \Gamma} K \cdot Y_k)^\vee$, where $\Gamma = \cup_{j=1}^d \mathrm{supp}_{\{Y_i\}}(N_j)$. Put

$$W = \{x \in V_{\mathrm{st}} \mid Y_j(x) = 0, \\ \forall Y_j \in \cup_{s \neq t}(\mathrm{supp}_{\{Y_i\}}(N_s) \cap \mathrm{supp}_{\{Y_i\}}(N_t))\}$$

and $H = \mathrm{Ker}(G \to GL(V_{\mathrm{st}}/W))$. Then (W, w) is a minimal paralleled linear full of (V, G) for some $w \in V_{\mathrm{st}}$ and $H = G_w$.

Example 2.7.2. Let ξ_j $(1 \leq j \leq n)$ denote the character of G^0 satisfying $(V^\vee)_{-\xi_j} \ni Y_j$. Suppose that $\dim G = 3$ express $\xi_j = \sum_{i=1}^3 c_{ij}\chi_i$ for some $c_{ij} \in \mathbf{Z}$, where $\{\chi_1, \chi_2, \chi_3\}$ generates the additive group $\mathfrak{X}(G^0)$. For some $(n >)$ $m \in \mathbf{N}$, assume that $\{j \mid 1 \leq j \leq m, c_{2j} < 0\} \neq \emptyset$,

$$\begin{cases} c_{2j} \leq 0,\ c_{3j} = 0\ ;\quad 1 \leq j \leq m \\ c_{1m+1} = c_{3m+1} = 0,\ c_{2m+1} = 1 \\ c_{3j} > 0\ ;\quad j > m+1 \end{cases}$$

and $\sharp(\{j \mid 1 \leq j \leq m, c_{1j} < 0\}) \cdot \sharp(\{j \mid 1 \leq j \leq m, c_{1j} > 0\}) \geq 2$. Then the condition on c_{3j}'s implies that

$$\mathcal{O}(V)^{\cap_{i=1,2}\mathrm{Ker}(\chi_i)} = K[Y_1, \dots, Y_{m+1}]^{\cap_{i=1,2}\mathrm{Ker}(\chi_i)}.$$

Since $\dim(G^0|_{\sum_{i=1}^{m+1} KY_i}) = 2$, we easily have

$$V_{\mathrm{st}} = \left(\sum_{j=1}^{m+1} KY_j\right)^\vee.$$

Putting

$$W = \{x \in V_{\mathrm{st}} \mid Y_{m+1}(x) = 0\}$$

and $H = G_{Y_{m+1}}$, we see that (W, w) is a minimal paralleled linear full of (V, G) for some $w \in V_{\mathrm{st}}$ and $H = G_w$.

Remark 2.7.3. We apply (2.6) to these examples as follows:

For the decomposition

$$\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_l \text{ (disjoint union)}$$

to non-empty subsets, put

$$H_{\{J_k\}} = \{\mathrm{diag}(c_1, \dots, c_m) \mid \forall c_j \in K \\ \text{such that } \prod_{j \in J_k} c_j = 1\ (1 \leq k \leq l)\}$$

defined on the the basis of W on which H is represented as a diagonal group. For a convenience sake, suppose that $\mathcal{R}_W(H)|_W = \{1\}$. Then $\mathcal{O}(V)^G$ is a Gorenstein ring if and only if $H|_W \subseteq H_{\{J_k\}}$ for some decomposition $\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_l$. In this case, a minimal subgroup $\widetilde{H} = H_{\{J_k\}}$ such that

$H|_W \subseteq H_{\{J_k\}}$ defines a canonical cofree embedding $(W, \widetilde{H})$ of (W, H).

3. Existence of cofree embeddings. For a homomorphism $A \to B$ of integral domains, let $\mathrm{Ht}_1(B, A)$ denote the set consisting of all prime ideals of B of height one whose restrictions to A are also of height one.

3.1. Cofree representations. Let (W, w) be a minimal paralleled linear hull of a finite dimensional representation (V, G) of a diagonalizable G. Let $\{X_1, \dots, X_m\}$ be a K-basis of the dual $W^\vee$ of W on which H is represented as a diagonal group, where H denotes G_w.

Let $D_{\{X_i\}}(W)$ be the subgroup of $GL(W)$ consisting of all elements which induces diagonal matrices on the K-basis $\{X_1, \dots, X_m\}$ of the dual module $W^\vee$. For a closed subgroup L of $GL(W)$ which is diagonal on $\{X_1, \dots, X_m\}$ such that (W, L) is stable, we note the following two facts:

Remark 3.1.1. For a prime ideal $\mathfrak{P} \in \mathrm{Ht}_1(\mathcal{O}(W), \mathcal{O}(W)^L)$, let $I_L(\mathfrak{P})$ denote the inertia group at $\mathfrak{P}$ and $\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathcal{O}(W)^L)$ the ramification index of $\mathfrak{P}$ over $\mathfrak{P} \cap \mathcal{O}(W)^L$ (cf. [N2]). Then we see that $I_L(\mathfrak{P})|_W$ is a finite group,

$$\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathcal{O}(W)^L) = \sharp(I_L(\mathfrak{P})|_W)$$

(e.g., [N2]) and there exists an element X_i in the set $\{X_1, \dots, X_m\}$ which principally generates $\mathfrak{P}$, for $\mathfrak{P}$ which is ramified over $\mathfrak{P} \cap \mathcal{O}(W)^L$ (i.e., $\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathcal{O}(W)^L) > 1$).

Lemma 3.1.2. *The following conditions* (1) *and* (2) *are equivalent for* (W, L):

(1) *The representation* (W, L) *is cofree.*

(2) *There exist the decomposition* $\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_l$ *(disjoint union) to nonempty subsets* J_i *and integers* $a_i \in \mathbf{N}$ $(1 \le i \le m)$ *such that*

$$\bigotimes_{j=1}^{l} \left(K\left[\prod_{i \in I_j} X_i^{a_i} \right] \right) = \mathcal{O}(W)^L.$$

In case where L is connected, the conditions (1) *and* (2) *are equivalent to*

(3) (W, L) *is equidimensional.*

Proof. In fact, suppose that (1) holds. Then (W, L^0) is equidimensional, which implies that (W, L^0) is cofree (cf. [W]). Thus there are a sum

$$\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_l \text{ (disjoint union)}$$

of non-empty subsets J_i and $b_i \in \mathbf{Z}_0$ $(1 \le i \le m)$ satisfying

$$\mathcal{O}(W)^{L^0} = \bigotimes_{j=1}^{l} \left(K\left[\prod_{i \in J_j} X_i^{b_i} \right] \right)$$

(cf. [W, N1]). By the complete reducibility, we see that $(W//L^0, L/L^0)$ is cofree. Since $W//L^0 \cong \mathbf{A}^l$ and L/L^0 is finite, the action of L/L^0 on the local ring of $W//L^0$ at the vertex induces an action of a finite group generated by pseudo-reflections on its Zariski tangent space (cf. [S1]). Hence the condition (2) holds. The implication (2) $\Rightarrow$ (1) can be easily shown. For the last assertion, see [W]. □

Lemma 3.1.3. *For any closed subgroups* D_i *of* $D_{\{X_i\}}(W)$ $(i = 1, 2)$, *if* $\mathcal{O}(W)^{D_1} = \mathcal{O}(W)^{D_2}$ *and* (W, D_1) *is stable, then* $D_1 = D_2$.

Proof. By the stability of (W, D_1), we see that

$$\prod_{i=1}^{m} X^{c_i} \in \mathcal{O}(W)^{D_1}$$

for some $c_i \in \mathbf{N}$ $(1 \le i \le m)$, which implies that (W, D_2) is also stable. Then, since $\mathcal{Q}(\mathcal{O}(W)^{D_i}) = \mathcal{Q}(\mathcal{O}(W))^{D_i}$, the assertion follows from the character theory of diagonalizable groups over the field K of characteristic zero. □

Lemma 3.1.4. *Let* $\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_l$ *(disjoint union) be the decomposition to non-empty subsets* J_i *and* $b_i \in \mathbf{N}$ $(1 \le i \le m)$ *any integers. Then there is a unique closed subgroup* D *in* $D_{\{X_i\}}(W)$ *such that*

$$\mathcal{O}(W)^D = \bigotimes_{j=1}^{l} \left(K\left[\prod_{i \in J_j} X_i^{b_i} \right] \right) \subseteq \mathcal{O}(W).$$

Proof. Let D denote the stabilizer of $D_{\{X_i\}}(W)$ at the set $\{\prod_{i \in J_j} X_i^{b_i} \mid 1 \le j \le l\}$ under the natural action of $D_{\{X_i\}}(W)$ on $\mathcal{O}(W)$. As

$$K\left[\prod_{i \in J_j} X_i^{b_i}, \middle| 1 \le j \le l \right] = K\left[\prod_{i \in J_j} X_i^{b_i}, 1/\prod_{i \in J_j} X_i^{b_i} \middle| 1 \le j \le l \right] \cap \mathcal{O}(W),$$

we must have

$$\mathcal{O}(W)^D = \bigotimes_{j=1}^{l} \left(K\left[\prod_{i \in J_j} X_i^{b_i} \right] \right) \subseteq \mathcal{O}(W).$$

On the other hand, denoting by D_j the stabilizer of $D_{\{X_i\}}(W)$ at the set $\{X_i \mid i \notin J_j\}$, we have

$$K[X_i \mid i \in J_j]^{D_j \cap D} = K\left[\prod_{i \in J_j} X_i^{b_i}\right].$$

Thus we must have

$$\mathcal{O}(W)^{(\prod_{j=1}^{l} D_j) \cap D} = \bigotimes_{j=1}^{l}\left(K\left[\prod_{i \in J_j} X_i^{b_i}\right]\right),$$

which implies $D = (\prod_{j=1}^{l} D_j) \cap D$. Consequently (W, D) is stable. The uniqueness of D follows from this and (3.1.3). □

Lemma 3.2. *Under the same circumstances as in the first paragraph in* (2.8), *let* $\widetilde{\phi} : \widetilde{H} \to GL(W)$ *be a faithful representation of a diagonalizable group* $\widetilde{H}$ *such that* $H|_W \subseteq \widetilde{H}|_W$. *In the case where the condition* (2) *in* (2.5) *holds, the last equality in* (1) *in* (2.5) *holds if and only if the canonical quotient morphism*

$$\pi_{W//H, \widetilde{H}/H} : W//H \to W//\widetilde{H}$$

is divisorially unramified (*for definition, cf.* [N2]).

Proof. Since $W \to W//H$ is no-blowing-up of codimension one (cf. (1.1)), we see

$$\mathrm{Ht}_1(\mathcal{O}(W), \mathcal{O}(W)^{\widetilde{H}}) \subseteq \mathrm{Ht}_1(\mathcal{O}(W), \mathcal{O}(W)^H).$$

On the other hand, for any $\mathfrak{Q} \in \mathrm{Ht}_1(\mathcal{O}(W), \mathcal{O}(W)^H)$ such that $I_H(\mathfrak{Q})|_W \neq \{1\}$, as in (2.8.1), we see that $\mathfrak{Q}$ is generated by the element of $W^\vee$ which is a relative invariant of $\widetilde{H}$. Thus, since $(W, \widetilde{H})$ is stable, the restriction $\mathfrak{Q} \cap \mathcal{O}(W)^{\widetilde{H}}$ is non-zero. The cofreeness of $(W, \widetilde{H})$ implies that $W \to W//\widetilde{H}$ is equidimensional, and so is $W//H \to W//\widetilde{H}$. Consequently, $\mathfrak{Q}$ is a member of $\mathrm{Ht}_1(\mathcal{O}(W), \mathcal{O}(W)^{\widetilde{H}})$. The equivalence in the assertion in (3.2) easily follows from the above observation and [N3]. □

Proof of (2.6). We use the notation in (3.1). The character

$$\mu_{\pi_{\mathcal{O}(W)\cdot X_i}} : I_H(\mathcal{O}(W) \cdot X_i) \ni \sigma \mapsto \sigma(X_i)/X_i \in \mathbf{U}(K)$$

can be identified with the restriction of $\det_{W^\vee}$. So, using the notation in (1.4) of [N3], we see

$$t_{\mathcal{O}(W)\cdot X_i}(\det_W|_H) = e_i - 1,$$

where $e_i = \sharp(I_H(\mathcal{O}(W) \cdot X_i)|_W)$. As $\mathcal{O}(W)^H \hookrightarrow \mathcal{O}(W)$ is no-blowing-up of codimension one (for definition, cf. [F]), we see that

$$\mathcal{O}(W)_{\det_W|_H} \cong \mathcal{O}(W)^H$$

if and only if

$$\prod_{i=1}^{l} X_i^{e_i - 1} \in \mathcal{O}(W)_{\det_W|_H}$$

(e.g., [N3]). If these conditions are satisfied, then

$$\mathcal{O}(W)_{\det_W|_H} = \mathcal{O}(W)^H \cdot \prod_{i=1}^{l} X_i^{e_i - 1}.$$

Clearly, since $\prod_{i=1}^{l} X_i \in \mathcal{O}(W)_{\det_{W^\vee}|_H}$, the affine variety $W//H$ is Gorenstein if and only if

$$\prod_{i=1}^{l} X_i^{e_i} \in \mathcal{O}(W)^H.$$

Suppose that the condition (2) holds, i.e., $(W, \widetilde{H})$ is a canonical cofree embedding of (W, H). Then, by (3.1.2), we have

$$\bigotimes_{j=1}^{l}\left(K\left[\prod_{i \in J_j} X_i^{a_i}\right]\right) = \mathcal{O}(W)^{\widetilde{H}}$$

for some decomposition

$\{1, \ldots, m = \dim W\} = J_1 \sqcup \cdots \sqcup J_l$ (disjoint union)

to nonempty subsets and $a_i \in \mathbf{N}$. Since

$$\mathrm{e}(\mathcal{O}(W) \cdot X_i, \mathcal{O}(W) \cdot X_i \cap \mathcal{O}(W)^{\widetilde{H}}) = a_i,$$

by (2.5) and (3.1.1), we must have

$$a_i = e_i (= \sharp(I_H(\mathcal{O}(W) \cdot X_i)|_W)),$$

which implies $\prod_{i=1}^{m} X_i^{e_i} \in \mathcal{O}(W)^{\widetilde{H}}$. From this and the observation of the former paragraph, we have just shown the condition (1) is satisfied.

Conversely, suppose that the condition (1) holds. Then

$$\prod_{i=1}^{m} X_i^{e_i} \in \mathcal{O}(W)^H \subseteq \mathcal{O}(W)^{\mathcal{R}_W(H)} = K[X_1^{e_1}, \ldots, X_l^{e_m}].$$

We can uniquely express this monomial as a product $\prod_{j=1}^{l} M_j$ of elements M_j's which are members of the unique minimal system of generators of $\mathcal{O}(W)^H$ consisting of monomials of $\{X_1, \ldots, X_m\}$. Obviously there exists the decomposition

$\{1, \ldots, m\} = J_1 \sqcup \cdots \sqcup J_l$ (disjoint union)

to nonempty subsets J_j such that $\prod_{i \in J_j} X_i^{e_i} = M_j$ $(1 \leq j \leq l)$. Let $\widetilde{H}$ be the stabilizer of $D_{\{X_i\}}(W)$

at $\{M_j \mid 1 \leq j \leq l\}$. By (3.1.4), the representation $(W, \widetilde{H})$ is stable and cofree. Since $\mathcal{O}(W)^H \hookrightarrow \mathcal{O}(W)$ is no-blowing-up of codimension one, applying (3.1.1) to (W, H) and $(W, \widetilde{H})$, we see that $W//H \to W//\widetilde{H}$ is divisorially unramified, which proves (2).

The remainder of the assertions in this theorem follows from (3.2) and the property of $\widetilde{H}$. □

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