Remarks on zeta functions and K-theory over $\mathbf{F}_1$

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Abstract: We show that the notion of zeta functions over the field of one element $\mathbf{F}_1$, as given in special cases by Soulé, extends naturally to all $\mathbf{F}_1$-schemes as defined by the author in an earlier paper. We further give two constructions of K-theory for affine schemes or $\mathbf{F}_1$-rings, we show that these coincide in the group case, but not in general.

Key words: Zeta function; field of one element.

Introduction. Soulé [10], inspired by Manin [7], gave a definition of zeta functions over the field of one element $\mathbf{F}_1$. We describe this definition as follows. Let X be a scheme of finite type over $\mathbf{Z}$. For a prime number p one sets after Weil,

$$Z_X(p,T) \stackrel{\text{def}}{=} \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \# X(\mathbf{F}_{p^n})\right),$$

where $\mathbf{F}_{p^n}$ denotes the field of p^n elements. This is the local zeta function over p, and the global zeta function of X is given as

$$\zeta_{X|\mathbf{Z}}(s) \stackrel{\text{def}}{=} \prod_p Z_X(p,p^{-s})^{-1}.$$

Soulé considered in [10] the following condition: Suppose there exists a polynomial $N(x)$ with integer coefficients such that $\#X(\mathbf{F}_{p^n}) = N(p^n)$ for every prime p and every $n \in \mathbf{N}$. Then $Z_X(p,p^{-s})^{-1}$ is a rational function in p and p^{-s}. The vanishing order at $p=1$ is $N(1)$. One may thus define

$$\zeta_{X|\mathbf{F}_1}(s) = \lim_{p\to 1} \frac{Z_X(p,p^{-s})^{-1}}{(p-1)^{N(1)}}.$$

One computes that if $N(x) = a_0 + a_1 x + \cdots + a_n x^n$, then

$$\zeta_{X|\mathbf{F}_1}(s) = s^{a_0}(s-1)^{a_1}\cdots(s-n)^{a_n}.$$

In the paper [1] there is given a definition of a scheme over $\mathbf{F}_1$ as well as an ascent functor $\cdot \otimes \mathbf{Z}$ from $\mathbf{F}_1$-schemes to $\mathbf{Z}$-schemes. An affine $\mathbf{F}_1$-scheme is given by a commutative monoid and its lift to $\mathbf{Z}$ is given by the corresponding monoidal ring. This procedure extends to general schemes as it respects gluing. We say that a $\mathbf{Z}$-scheme is *defined over* $\mathbf{F}_1$, if it comes by ascent from a scheme over $\mathbf{F}_1$. The natural question arising is whether schemes defined over $\mathbf{F}_1$ satisfy Soulé's condition.

Simple examples show that this is not the case. However, schemes defined over $\mathbf{F}_1$ satisfy a slightly weaker condition which serves the purpose of defining $\mathbf{F}_1$-zeta functions as well, and which we give in the following theorem.

Theorem 1. *Let X be a $\mathbf{Z}$-scheme of finite type defined over $\mathbf{F}_1$. Then there exists a natural number e and a polynomial $N(x)$ with integer coefficients such that for every prime power q one has*

$$(q-1,e) = 1 \quad \Rightarrow \quad \#X(\mathbf{F}_q) = N(q).$$

This condition determines the polynomial N uniquely (independent of the choice of e). We call it the zeta-polynomial *of X.*

With this theorem, we can define the zeta function of an arbitrary $\mathbf{F}_1$-scheme X as

$$\zeta_{X|\mathbf{F}_1}(s) = s^{a_0}(s-1)^{a_1}\cdots(s-n)^{a_n},$$

if $N_X(x) = a_0 + a_1 x + \cdots a_n x^n$ is its zeta-polynomial.

We also define its *Euler characteristic* as

$$\chi(X) = N_X(1) = a_1 + \cdots + a_n.$$

This definition is due to Soulé [10]. We repeat the justification given in [6], which is based on the Weil conjectures.

Suppose that $X/\mathbf{F}_p = X_Z \times_{\mathbf{Z}} \mathbf{F}_p$ is a smooth projective variety over the finite field $\mathbf{F}_p$. Then the Weil conjectures, as proven by Deligne, say that

$$Z_{X_{\mathbf{Z}}}(p,T) = \prod_{l=0}^{m} P_l(T)^{(-1)^{l+1}},$$

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with

$$P_l(T) = \prod_{j=1}^{b_l}(1 - \alpha_{l,j}T),$$

satisfying $|\alpha_{l,j}| = p^{l/2}$, where b_l is the l-th Betti-number.

On the other hand, suppose that $\#X(\mathbf{F}_{p^n}) = N(p^n)$ holds for every $n \in \mathbf{N}$, where $N(x) = a_0 + a_1x + \cdots + a_nx^n$ is the zeta-polynomial, then one gets

$$Z_{X_\mathbf{Z}}(p,T) = \prod_{k=0}^{n}(1 - p^kT)^{-a_k}.$$

Comparing these two expressions, one gets

$$b_l = \begin{cases} a_{l/2} & l \text{ even}, \\ 0 & l \text{ odd}. \end{cases}$$

So $\sum_{k=0}^n a_k = \sum_{l=0}^m (-1)^l b_l$ is the Euler characteristic.

For explicit computations of zeta functions and Euler numbers over $\mathbf{F}_1$ as defined above, see [6], where there are given examples of varieties satisfying Soulé's condition. Not all of them, though, come from $\mathbf{F}_1$.

Next for K-theory. Based on the idea of Tits, that $\mathrm{GL}_n(\mathbf{F}_1)$ should be the permutation group $\mathrm{Per}(n)$, Soulé also suggested that

$$K_i(\mathbf{F}_1) = \pi_i(B(\mathrm{Per}(\infty))^+),$$

which is known to coincide with the stable homotopy group of the spheres, $\pi_i^s = \lim_{k\to\infty} \pi_{i+k}(S^k)$. (The + refers to Quillen's + construction.) More generally, for a monoid A, or an $\mathbf{F}_1$-ring $\mathbf{F}_A$, one has

$$\mathrm{GL}_n(A) = \mathrm{GL}_n(\mathbf{F}_A) = (A^\times)^n \rtimes \mathrm{Per}(n),$$

where $A^\times$ is the group of units in A. Setting $\mathrm{GL}(A) = \lim_{n\to\infty} \mathrm{GL}_n(A)$, one lets

$$K_i^+(A) = \pi_i(BGL(A)^+).$$

On the other hand, one considers the category $\mathcal{P}$ of all finitely generated projective modules over A and defines

$$K_i^Q(A) = \pi_{i+1}(BQ\mathcal{P}),$$

where Q means Quillen's Q-construction. It turns out that $\pi_1(BQ\mathcal{P})$ coincides with the Grothendieck group $K_0(\mathcal{P})$ of $\mathcal{P}$. If A is a group, these two definitions of K-theory agree, but not in general.

A calculation shows, that if A is an abelian group, then

$$K_i(A) = \begin{cases} \mathbf{Z} \times A & i = 0, \\ \pi_i^s & i > 0. \end{cases}$$

So, for general A, since one has $K^+(A) = K^+(A^\times)$, this identity completely computes K^+. Furthermore, for every A one has a canonical homomorphism $K_i^+(A) \to K_i^Q(A)$.

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1. $\mathbf{F_1}$-schemes. For basics on $\mathbf{F}_1$-schemes we refer to [1].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An *ideal* $\mathfrak{a}$ of a monoid A is a subset with $A\mathfrak{a} \subset \mathfrak{a}$. A *prime ideal* is an ideal $\mathfrak{p}$ such that $S_\mathfrak{p} = A \setminus \mathfrak{p}$ is a submonoid of A. For a prime ideal $\mathfrak{p}$ let $A_\mathfrak{p} = S_\mathfrak{p}^{-1}A$ be the *localisation* at $\mathfrak{p}$. The *spectrum* of a monoid A is the set of all prime ideals with the obvious Zariski-topology (see [1]). Similar to the theory of rings, one defines a structure sheaf $\mathcal{O}_X$ on $X = \mathrm{spec}(A)$, and one defines a *scheme over* $\mathbf{F}_1$ to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A $\mathbf{F}_1$-scheme X is *of finite type*, if it has a finite covering by affine schemes $U_i = \mathrm{spec}(A_i)$ such that each A_i is finitely generated.

For a monoid A we let $A \otimes \mathbf{Z}$ be the monoidal ring $\mathbf{Z}[A]$. This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring R to the multiplicative monoid $(R, \times)$. This construction is compatible with gluing, so one gets a functor $X \mapsto X_\mathbf{Z}$ from $\mathbf{F}_1$-schemes to $\mathbf{Z}$-schemes.

Lemma 2. *X is of finite type if and only if $X_\mathbf{Z}$ is a $\mathbf{Z}$-scheme of finite type.*

Proof. If X is of finite type, it is covered by finitely many affines $\mathrm{spec}(A_i)$, where A_i is finitely generated, hence $\mathbf{Z}[A_i]$ is finitely generated as a $\mathbf{Z}$-algebra and so it follows that $X_\mathbf{Z}$ is of finite type.

Now suppose that $X_\mathbf{Z}$ is of finite type. Consider a covering of X by open sets of the form $U_i = \mathrm{spec}(A_i)$. then one gets an open covering of $X_\mathbf{Z}$ by sets of the form $\mathrm{spec}(\mathbf{Z}[A_i])$, with the spectrum in the ring-sense. Since $X_\mathbf{Z}$ is compact, we may assume this covering finite. As $X_\mathbf{Z}$ is of finite type, each $\mathbf{Z}[A_i]$ is a finitely generated $\mathbf{Z}$-algebra. Let S be a generating set of A_i. Then it generates $\mathbf{Z}[A_i]$, and hence it contains a finite generating set T of $\mathbf{Z}[A_i]$. Then T also generates A_i as a monoid, so A_i is finitely generated. □

2. Proof of Theorem 1. We will show uniqueness first.

Lemma 3. *For every natural number e there are infinitely many prime powers q with $(q-1, e) = 1$.*

Proof. Write $e = 2^k m$ where m is odd. Let $n \in \mathbf{N}$. The number 2^n is a unit modulo m and hence there are infinitely many n such that $2^n \equiv 1$ modulo m. Replacing n by $n+1$ we see that there are infinitely many n such that $2^n \equiv 2$ modulo m and hence $2^n - 1 \equiv 1$ modulo m. As $2^n - 1$ is odd, it follows $(2^n - 1, e) = 1$ for every such n. □

Now for the uniqueness of N. Suppose that the pairs (e, N) and (e', N') both satisfy the theorem. Then for every prime power q one has

$$(q-1, ee') = 1 \quad \Rightarrow \quad N(q) = \#X(\mathbf{F}_q) = N'(q).$$

As there are infinitely many such prime powers q, it follows that $N(x) = N'(x)$, as claimed.

We start on the existence of N. For a finite abelian group E define its *exponent* $m = \exp(E)$ to be the smallest number m such that $x^m = 1$ for every $x \in G$. The exponent is the least common multiple of the orders of elements of G. A finitely generated abelian group G is of the form $\mathbf{Z}^r \times E$ for a finite group E. Then r is called the *rank* of G and the exponent of E is called the *exponent* of G.

For a finitely generated monoid A we denote by $\mathrm{Quot}(A)$ its quotient group. This group comes about by inverting every element in A. It has a natural morphism $A \to \mathrm{Quot}(A)$ and the universal property that every morphism from A to a group factorizes uniquely over $A \to \mathrm{Quot}(A)$. In the language of [1], $\mathrm{Quot}(A)$ coincides with the stalk $\mathcal{O}_\eta = A_\eta$ at the generic point η of $\mathrm{spec}(A)$.

We define the *rank* and *exponent* to be the rank and exponent of $\mathrm{Quot}(A)$. Note that for a finitely generated monoid A the spectrum $\mathrm{spec}(A)$ is a finite set. Hence the underlying space of a scheme X over $\mathbf{F}_1$ of finite type is a finite set. We then define the exponent of X to be the least common multiple of the numbers $\exp(\mathcal{O}_\mathfrak{p})$, where $\mathfrak{p}$ runs through the finite set X.

Let X be a scheme over $\mathbf{F}_1$ of finite type. We may assume that X is connected. Let e be its exponent. Let q be a prime power and let D_q be the monoid $(\mathbf{F}_q, \times)$. Then $\#X_\mathbf{Z}(\mathbf{F}_q) = \#X(D_q)$, where $X(D) = \mathrm{Hom}(D, X)$ as usual. For an integer $k \geq 2$ let C_{k-1} denote the cyclic group of $k-1$ elements and let D_k be the monoid $C_{k-1} \cup \{0\}$, where $x \cdot 0 = 0$. Note that if q is a prime power, then $D_q \cong (\mathbf{F}_q, \times)$, where $\mathbf{F}_q$ is the field of q elements.

Fix a covering of X by affines $U_i = \mathrm{spec} A_i$. Since $\mathrm{spec}(D_k)$ consists of two points, the generic, which always maps to the generic point and the closed point, it follows that

$$X(\mathrm{spec}(D_k)) = \bigcup_i U_i(\mathrm{spec}(D_k)),$$

and thus the cardinality of the right hand side may be written as an alternating sum of terms of the form

$$\#U_{i_1} \cap \cdots \cap U_{i_s}(\mathrm{spec}(D_k)).$$

Now $U_{i_1} \cap \cdots \cap U_{i_s}$ is itself a union of affines and so this term again becomes an alternating sum of similar terms. This process stops as X is a finite set. Therefore, to prove the theorem, it suffices to assume that X is affine.

So we assume that $X = \mathrm{spec}(A)$ for a finitely generated monoid A. In this case $X(\mathrm{spec}(D_k)) = \mathrm{Hom}(A, D_k)$. For a given monoid morphism $\varphi : A \to D_k$ we have that $\varphi^{-1}(\{0\})$ is a prime ideal in A, call it $\mathfrak{p}$. Then φ maps $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$ to the group C_{k-1}. So $\mathrm{Hom}(A, D_k)$ may be identified with the disjoint union of the sets $\mathrm{Hom}(S_\mathfrak{p}, C_{k-1})$ where $\mathfrak{p}$ ranges over $\mathrm{spec}(A)$. Now C_{k-1} is a group, so every homomorphism from $S_\mathfrak{p}$ to C_{k-1} factorises over the quotient group $\mathrm{Quot}(S_\mathfrak{p})$ and one gets $\mathrm{Hom}(S_\mathfrak{p}, C_{k-1}) = \mathrm{Hom}(\mathrm{Quot}(S_\mathfrak{p}), C_{k-1})$. Note that $\mathrm{Quot}(S_\mathfrak{p})$ is the group of units in the stalk $\mathcal{O}_{X,\mathfrak{p}}$ of the structure sheaf, therefore does not depend on the choice of the affine neighbourhood. The group $\mathrm{Quot}(S_\mathfrak{p})$ is a finitely generated abelian group. Let r be its rank and e its exponent. If e is coprime to $k-1$, then there is no non-trivial homomorphism from the torsion part of $\mathrm{Quot}(S_\mathfrak{p})$ to C_{k-1} and so in that case $\#\mathrm{Hom}(S_p, C_{k-1}) = (k-1)^r$. This proves the existence of e and N and finishes the proof of Theorem 1. □

Remark 1. We have indeed proved more than Theorem 1. For an $\mathbf{F}_1$-scheme X of finite type we define $X(\mathbf{F}_q) = \mathrm{Hom}(\mathrm{spec}(\mathbf{F}_q), X)$, where the Hom takes place in the category of $\mathbf{F}_1$-schemes, and $\mathbf{F}_q$ stands for the multiplicative monoid of the finite field. It follows that

$$X(\mathbf{F}_q) \ \cong \ X_\mathbf{Z}(\mathbf{F}_q).$$

Further, for $k \in \mathbf{N}$ one sets $\mathbf{F}_k = D_k$ then this notation is consistent and we have proved above,

$$(k-1, e) = 1 \quad \Rightarrow \quad \#X(\mathbf{F}_k) = N(k),$$

where e now is a well defined number, the exponent of X. Further it follows from the proof, that the degree of N is at most equal to the rank of X, which is defined as the maximum of the ranks of the local monoids $\mathcal{O}_\mathfrak{p}$, for $\mathfrak{p} \in X$.

Remark 2. As the proof of Theorem 1 shows, the zeta-polynomial N_X of X, does actually not depend on the structure sheaf $\mathcal{O}_X$, but on the subsheaf of units $\mathcal{O}_X^\times$, where for every open set U in X the set $\mathcal{O}_X^\times(U)$ is defined to be the set of sections $s \in \mathcal{O}_X(U)$ such that $s(p)$ lies in $\mathcal{O}_{X,p}^\times$ for every $p \in U$. We therefore call $\mathcal{O}_X^\times$ the *zeta sheaf* of X.

3. K-theory. In this section we give two definitions of K-theory over $\mathbf{F}_1$ and we show that they do coincide for groups, but not in general. This approach follows Quillen [9].

3.1. The +-construction. Let A be a monoid. Recall from [1] that $\mathrm{GL}_n(A)$ is the group of all $n \times n$ matrices with exactly one non-zero entry in each row and each column, and this entry being an element of the unit group $A^\times$. We also write $A^\times$ as the stalk A_c at the closed point c of spec(A). In other words, we have

$$\mathrm{GL}_n(A) \;\cong\; A_c^n \rtimes \mathrm{Per}(n),$$

where $\mathrm{Per}(n)$ is the permutation group in n letters, acting on A_c^n by permuting the co-ordinates.

There is a natural embedding $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ by setting the last co-ordinate equal to 1. We define the group

$$\mathrm{GL}(A) \stackrel{\mathrm{def}}{=} \varinjlim_n \mathrm{GL}_n(A).$$

Similar to the K-theory of rings [9] for $j \geq 0$ we define

$$K_j^+(A) \stackrel{\mathrm{def}}{=} \pi_j(\mathrm{BGL}(A)^+),$$

where $\mathrm{BGL}(A)$ is the classifying space of $\mathrm{GL}(A)$, the $+$ means the +-construction, and π_j is the j-th homotopy group. For instance, $K_j^+(\mathbf{F}_1)$ is the j-th stable homotopy group of the spheres [8].

3.2. The Q-construction. A category is called *balanced*, if every morphism which is epi and mono, already has an inverse, i.e., is an isomorphism.

Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is called *injective* if for every monomorphism $M \hookrightarrow N$ the induced map $\mathrm{Mor}(N, I) \to \mathrm{Mor}(M, I)$ is surjective. Conversely, an object $P \in \mathcal{C}$ is called *projective* if for every epimorphism $M \twoheadrightarrow N$ the induced map $\mathrm{Mor}(P, M) \to \mathrm{Mor}(P, N)$ is surjective. We say that $\mathcal{C}$ *has enough injectives* if for every $A \in \mathcal{C}$ there exists a monomorphism $A \hookrightarrow I$, where I is an injective object. Likewise, we say that $\mathcal{C}$ *has enough projectives* if for every $A \in \mathcal{C}$ there is an epimorphism $P \twoheadrightarrow A$ with P projective.

A category $\mathcal{C}$ is *pointed* if it has an object 0 such that for every object X the sets $\mathrm{Mor}(X, 0)$ and $\mathrm{Mor}(0, X)$ have exactly one element each. The zero object is uniquely determined up to unique isomorphy. In every set $\mathrm{Mor}(X, Y)$ there exists a unique morphism which factorises over the zero object, this is called the zero morphism. In a pointed category it makes sense to speak of kernels and cokernels. Kernels are always mono and cokernels are always epimorphisms. A sequence

$$0 \longrightarrow X \xrightarrow{\;i\;} Y \xrightarrow{\;j\;} Z \longrightarrow 0$$

is called *strong exact*, if i is the kernel of j and j is the cokernel of i. We say that the sequence *splits*, if it is isomorphic to the natural sequence

$$0 \;\to\; X \;\to\; X \oplus Z \;\to\; Z \;\to\; 0.$$

Assume that kernels and cokernels always exist. Then every kernel is the kernel of its cokernel and every cokernel is the cokernel of its kernel. For a morphism f let $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$ and $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$. If $\mathcal{C}$ has enough projectives, then the canonical map $\mathrm{im}(f) \to \mathrm{coim}(f)$ has zero kernel and if $\mathcal{C}$ has enough injectives, then this map has zero cokernel.

Let $\mathcal{C}$ be a pointed category and $\mathcal{E}$ a class of strong exact sequences. The class $\mathcal{E}$ is called *closed under isomorphism*, or simply *closed* if every sequence isomorphic to one in $\mathcal{E}$, lies in $\mathcal{E}$. Every morphism occurring in a sequence in $\mathcal{E}$ is called an $\mathcal{E}$-morphism.

A balanced pointed category $\mathcal{C}$, together with a closed class $\mathcal{E}$ of strong exact sequences is called a *quasi-exact category* if

- for any two objects X, Y the natural sequence

  $$0 \to X \to X \oplus Y \to Y \to 0$$

  belongs to $\mathcal{E}$,
- the class of $\mathcal{E}$-kernels is closed under composition and base-change by $\mathcal{E}$-cokernels, likewise, the class of $\mathcal{E}$-cokernels is closed under composition and base change by $\mathcal{E}$-kernels.

Let $(\mathcal{C}, \mathcal{E})$ be a quasi-exact category. We define the category $Q\mathcal{C}$ to have the same objects as $\mathcal{C}$, but a morphism from X to Y in $Q\mathcal{C}$ is an isomorphism class of diagrams of the form

$$\begin{array}{ccc} S & \hookrightarrow & Y \\ \downarrow\!\!\!\downarrow & & \\ X, & & \end{array}$$

where the horizontal map is a $\mathcal{E}$-kernel in $\mathcal{C}$ and the vertical map is a $\mathcal{E}$-cokernel. The composition of two Q-morphisms

$$\begin{array}{ccc} S & \hookrightarrow & Y \\ \downarrow\!\!\!\downarrow & & \\ X, & & \end{array} \qquad \begin{array}{ccc} T & \hookrightarrow & Z \\ \downarrow\!\!\!\downarrow & & \\ Y, & & \end{array}$$

is given by the base change $S \times_Y T$ as follows:

$$\begin{array}{ccccc} S \times_Y T & \hookrightarrow & T & \hookrightarrow & Z \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\ S & \hookrightarrow & Y & & \\ \downarrow\!\!\!\downarrow & & & & \\ X. & & & & \end{array}$$

Every $\mathcal{E}$-kernel $i : X \hookrightarrow Y$ gives rise to a morphism $i_!$ in $Q\mathcal{C}$, and every $\mathcal{E}$-cokernel $p : Z \twoheadrightarrow X$ gives rise to a morphism $p^! : X \to Z$ in $Q\mathcal{C}$. By definition, every morphism in $Q\mathcal{C}$ factorises as $i_! p^!$ uniquely up to isomorphism.

Let $(\mathcal{C}, \mathcal{E})$ be a small quasi-exact category. Then the classifying space $BQ\mathcal{C}$ is defined. Note that for every object X in $Q\mathcal{C}$ there is a morphism from 0 to X, so that $BQ\mathcal{C}$ is path-connected. We consider the fundamental group $\pi_1(BQ\mathcal{C})$ as based at a zero 0 of $\mathcal{C}$.

Theorem 4. *The fundamental group $\pi_1(BQ\mathcal{C})$ is canonically isomorphic to the Grothendieck group $K_0(\mathcal{C}) = K_0(\mathcal{C}, \mathcal{E})$.*

Proof. This proof is taken from [9], where it is done for exact categories, we repeat it for the convenience of the reader. The Grothendieck group $K_0(\mathcal{C}, \mathcal{E})$ is the abelian group with one generator $[X]$ for each object X of $\mathcal{C}$ and a relation $[X] = [Y][Z]$ for every strong exact sequence

$$0 \longrightarrow Y \hookrightarrow X \twoheadrightarrow Z \longrightarrow 0$$

in $\mathcal{E}$. According to Proposition 1 of [9], it suffices to show that for a morphism-inverting functor $F : Q\mathcal{C} \to \text{Sets}$ the group $K_0(\mathcal{C})$ acts naturally on $F(0)$ and that the resulting functor from the category $\mathcal{F}$ of all such F to $K_0(\mathcal{C})$-sets is an equivalence of categories.

For $X \in \mathcal{C}$ let i_X denote the zero kernel $0 \to X$, and let j_X be the zero cokernel $X \to 0$. Let $\mathcal{F}'$ be the full subcategory of $\mathcal{F}$ consisting of all F such that $F(X) = F(0)$ and $F(i_{X!}) = \mathrm{id}_{F(0)}$ for every X. Any $F \in \mathcal{F}$ is isomorphic to an object of $\mathcal{F}'$, so it suffices to show that $\mathcal{F}'$ is equivalent to $K_0(\mathcal{C})$-sets. So let $F \in \mathcal{F}'$, for a kernel $i : X \hookrightarrow Y$ we have $i i_X = i_Y$, so that $F(i_!) = \mathrm{id}_{F(0)}$. Given a strong exact sequence

$$0 \longrightarrow X \overset{i}{\hookrightarrow} Y \overset{j}{\twoheadrightarrow} Z \longrightarrow 0,$$

we have $j^! i_{Z!} = i_! j^!_X$, hence $F(j^!) = F(j^!_X) \in \mathrm{Aut}(F(0))$. Also,

$$F(j^!_Y) = F(j^! j^!_Z) = F(j^!_X) F(j^!_Z).$$

So by the universal property of $K_0(\mathcal{C})$, there is a unique homomorphism from $K_0(\mathcal{C})$ to $\mathrm{Aut}(F(0))$ such that $[X] \mapsto F(j^!_X)$. So we have a natural action of $K_0(\mathcal{C})$ on $F(0)$, hence a functor from $\mathcal{F}'$ to $K_0(\mathcal{C})$-sets given by $F \mapsto F(0)$.

The other way around let S be a $K_0(\mathcal{C})$-set, and let $F_S : Q\mathcal{C} \to \text{Sets}$ be the functor defined by $F_S(X) = S$, $F_S(i_! j^!) =$ multiplication by $[\ker j]$ on S. To see that this is indeed a functor, it suffices to show that $F_S(j^! i_!) = F_S(j^!)$. It holds $j^! i_! = i_{1!} j_1^!$, where i_1 and j_1 are given by the cartesian diagram

$$\begin{array}{ccc} A & \overset{i_1}{\hookrightarrow} & X \\ {\scriptstyle j_1}\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow{\scriptstyle j} \\ Z & \overset{i}{\hookrightarrow} & Y. \end{array}$$

It follows $F_S(j^! i_!) = F_S(i_{1!} j_1^!) = [\ker j_1]$. Using the cartesian diagram one sees that $\ker j_1$ is isomorphic to $\ker j$. It is easy to verify that the two functors given are inverse to each other up to isomorphism, whence the theorem. □

This theorem motivates the following definition,

$$K_i(\mathcal{C}, \mathcal{E}) \overset{\text{def}}{=} \pi_{i+1}(BQ\mathcal{C}).$$

For a monoid A we let $\mathcal{P}$ be the category of finitely generated pointed projective A-modules, or rather a small category equivalent to it, and we set

$$K_i^Q(A) \overset{\text{def}}{=} K_i(\mathcal{P}, \mathcal{E}),$$

where $\mathcal{E}$ is the class of sequences in $\mathcal{P}$ which are strong exact in the category of all modules. These sequences all split, which establishes the axioms for a quasi-exact category.

The two K-theories we have defined, do not coincide. For instance for the monoid of one generator $A = \{1, a\}$ with $a^2 = a$ one has

$$K_0^+(A) = \mathbf{Z}, \qquad K_0^Q(A) = \mathbf{Z} \times \mathbf{Z}.$$

The reason for this discrepancy is that $K_i^+(A)$ only depends on the group of units $A^\times$, but $K_i^Q(A)$ is sensible to the whole structure of A. So these two K-theories are unlikely to coincide except when A is a group, in which case they do, as the last theorem of this paper shows,

Theorem 5. *If A is an abelian group, then $K_i^+(A) = K_i^Q(A)$ for every $i \geq 0$.*

Proof. For a group each projective module is free, hence the proof of Grayson [3] of the corresponding fact for rings goes through. □

So, if A is a group, this defines $K_i(A)$ unambiguously. In particular, computations of Priddy [8] show that $K_i(\mathbf{F}_1) = \pi_s^i$ is the i-th stable homotopy group of the spheres. If A is a group, then every projective module is free. Based on this, one can use the Q-construction to show that if A is an abelian group, then

$$K_i(A) = \begin{cases} \mathbf{Z} \times A & i = 0, \\ \pi_i^s & i > 0. \end{cases}$$

This is proved as follows: As A is a group, a module over A is projective iff it is free. Therefore the category $\mathcal{P}$ is the product of the category Set_0 of pointed sets and A (considered as a category with one object). So $BQ\mathcal{P}$ is the product of $BQ\mathrm{Set}_0$ and $BQA = BA$, the classifying space of the group A. This implies the above result for $K_i(A)$.

For an arbitrary monoid A we conclude that $K_i^+(A) = K_i^+(A^\times) = K_i(A^\times)$, which we now can express in terms of the stable homotopy groups π_i^s.

Further, for every A one has a canonical homomorphism $K_i^+(A) \to K_i^Q(A)$ given by the map $K^Q(A^\times) \to K^Q(A)$. The latter comes about by the fact that every projective $A^\times$-module is free. Note that general functoriality under monoid homomorphism is granted for K^+, but not for K^Q. This contrasts the situation of rings, and has its reason in the fact that not every projective is a direct summand of a free module.

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