Smooth toric Fano five-folds of index two

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Abstract: In this paper, we classify smooth toric Fano 5-folds of index 2. There exist exactly 10 smooth toric Fano 5-folds of index 2 up to isomorphisms.

Key words: Toric variety; Fano variety; Fano index.

1. Introduction. For a smooth Fano d-fold X, the *index* i_X of X is defined as follows:

$$i_X := \max\{m \in \mathbf{Z}_{\geq 1} \mid -K_X = mH \text{ for a divisor } H\}.$$

There is a famous result of [KO] which says $1 \leq i_X \leq d+1$ and a smooth Fano d-fold of index $d+1$ or d is isomorphic to $\mathbf{P}^d$ or Q^d, respectively, where Q^d is the d-dimensional quadric. We remark that $Q^2 \cong \mathbf{P}^1 \times \mathbf{P}^1$ is the only case where a quadric hypersurface is isomorphic to a toric variety (see e.g. [Fjn]). A smooth Fano d-fold of index $d-1$ or $d-2$ is called a *del Pezzo* manifold or a *Mukai* manifold, respectively, and there are classifications for these manifolds (see [Fjt, Me, Mu]).

So, the next problem is the classification of smooth Fano d-folds of index $d-3$. If $d \geq 6$ and the Picard number is greater than 1, there is the classification (see [W]). For the case $d = 5$, there are some partial classifications (see [CO, NO]). Toward the general classification, in this paper, we classify smooth toric Fano 5-folds of index 2. We show that there exist exactly 10 smooth toric Fano 5-folds of index 2 (see Theorem 3.6). We remark that since a smooth complete toric d-fold of Picard number 1 is isomorphic to $\mathbf{P}^d$, this result completes the classification of smooth toric Fano d-folds of index $d-3$.

The content of this paper is as follows: Section 2. is a section for preparation. We review the Mori theory for smooth toric varieties. In Section 3., we consider the classification of smooth toric Fano 5-folds of index 2. Section 4. is devoted to constructing examples of toric Fano manifolds of index 2 which have no projective space bundle structure.

2. Preliminaries. In this section, we explain some basic facts of the toric geometry. See [Ba1, Ba2, FS, Fl, O, S1] for the detail.

Let Σ be a nonsingular complete fan in $N := \mathbf{Z}^d$, $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ and $X = X_\Sigma$ the associated smooth complete toric d-fold over an algebraically closed field k. Let $\mathrm{G}(\Sigma)$ be the set of primitive generators of 1-dimensional cones in Σ. A subset $P \subset \mathrm{G}(\Sigma)$ is called a *primitive collection* if P does not generate a cone in Σ, while any proper subset of P generates a cone in Σ. We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ. For a primitive collection $P = \{x_1, \ldots, x_m\}$, there exists the unique cone $\sigma(P)$ in Σ such that $x_1 + \cdots + x_m$ is contained in its relative interior since Σ is complete. So, we obtain an equality

$$(1) \qquad x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n,$$

where $y_1, \ldots, y_n$ are the generators of $\sigma(P)$, that is, $\sigma(P) \cap \mathrm{G}(\Sigma) = \{y_1, \ldots, y_n\}$, and $b_1, \ldots, b_n$ are positive integers. We call this equality the *primitive relation* of P. By the standard exact sequence

$$0 \to M \to \mathbf{Z}^{\mathrm{G}(\Sigma)} \to \mathrm{Pic}(X) \to 0$$

for a smooth toric variety, we have

$$A_1(X) \simeq \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(X), \mathbf{Z}) \simeq \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}/M, \mathbf{Z}) \\ \simeq M^{\perp} \subset \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}, \mathbf{Z}),$$

where $A_1(X)$ is the group of 1-cycles on S modulo rational equivalences, and hence

$$A_1(X) \simeq \left\{ (b_x)_{x \in \mathrm{G}(\Sigma)} \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{\mathrm{G}(\Sigma)}, \mathbf{Z}) \;\middle|\; \sum_{x \in \mathrm{G}(\Sigma)} b_x x = 0 \right\}.$$

Thus, by the equality $x_1 + \cdots + x_m - (b_1 y_1 + \cdots + b_n y_n) = 0$, we obtain an element $r(P)$ in $A_1(X)$ for

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each primitive collection $P \in \mathrm{PC}(\Sigma)$. We define the *degree* of P as $\deg P := (-K_X \cdot r(P)) = m - (b_1 + \cdots + b_n)$.

Proposition 2.1 ([Ba1, C, R]). *Let $X = X_\Sigma$ be a smooth projective toric variety. Then, the Mori cone of X is described as*

$$\mathrm{NE}(X) = \sum_{P \in \mathrm{PC}(\Sigma)} \mathbf{R}_{\geq 0} r(P) \subset A_1(X) \otimes \mathbf{R}.$$

A primitive collection P is said to be *extremal* if $r(P)$ is contained in an extremal ray of $\mathrm{NE}(X)$.

Remark 2.2. If $x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n$ is an extremal primitive relation, then $m + n \leq d + 1$, because $r(P)$ corresponds to an irreducible torus invariant curve.

Corollary 2.3. *Let $X = X_\Sigma$ be a smooth projective toric variety. Then, X is* Fano *if and only if $\deg P > 0$ for any extremal primitive collection $P \in \mathrm{PC}(\Sigma)$.*

For extremal primitive relations, we need the following propositons and definition for the classification.

Proposition 2.4 ([C, S1]). *Let $X = X_\Sigma$ be a smooth projective toric variety and P an extremal primitive collection. Then, for any $P' \in \mathrm{PC}(\Sigma) \backslash \{P\}$ such that $P \cap P' \neq \emptyset$,*

$$(P \setminus P') \cup (\sigma(P) \cap \mathrm{G}(\Sigma))$$

contains a primitive collection.

Proposition 2.5 ([C, Proposition 4.3]). *Let $X = X_\Sigma$ be a toric Fano manifold and P a primitive collection. If $\deg P \leq \deg P'$ for any primitive collection $P' \in \mathrm{PC}(\Sigma)$, then P is extremal.*

Definition 2.6 ([Ba1]). *Let $X = X_\Sigma$ be a smooth projective toric variety. Then, Σ is a* splitting *fan if $P \cap P' = \emptyset$ for any P, $P' \in \mathrm{PC}(\Sigma)$ such that $P \neq P'$.*

If Σ is a splitting fan, then there exists a sequence of smooth complete toric varieties

$$X = X_\Sigma =: X_s \stackrel{\varphi_s}{\to} X_{s-1} \stackrel{\varphi_{s-1}}{\to} \cdots \stackrel{\varphi_3}{\to} X_2 \stackrel{\varphi_2}{\to} X_1 \simeq \mathbf{P}^l,$$

where $X_i \stackrel{\varphi_i}{\to} X_{i-1}$ is a toric projective space bundle and $l \in \mathbf{Z}_{\geq 1}$. We remark that s is the Picard number of X. The number of the primitive collections of Σ is also s.

3. Classification. We start the classification.

Let $X = X_\Sigma$ be a smooth toric fano 5-fold of index 2. In this case, $\deg P$ is an even number for any $P \in \mathrm{PC}(\Sigma)$. Then, by Remark 2.2 and Corollary 2.3, the type of any extremal primitive relation is one of the following:

(1) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$,

(2) $x_1 + x_2 + x_3 + x_4 + x_5 = y_1$,

(3) $x_1 + x_2 + x_3 + x_4 + x_5 = 3y_1$,

(4) $x_1 + x_2 + x_3 + x_4 = 0$,

(5) $x_1 + x_2 + x_3 + x_4 = 2y_1$,

(6) $x_1 + x_2 + x_3 + x_4 = y_1 + y_2$,

(7) $x_1 + x_2 + x_3 = y_1$ and

(8) $x_1 + x_2 = 0$,

where $\{x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2\} \subset \mathrm{G}(\Sigma)$.

First of all, the existence of an extremal primitive relation of type (1) imply that $X \cong \mathbf{P}^5$, but $\mathbf{P}^5$ is of index 6. So, there does not exist an extremal primitive relation of type (1).

Proposition 3.1. *Let $X = X_\Sigma$ be a smooth toric Fano 5-fold of index 2. If X has an extremal primitive relation of type (2) or (3), then X is isomorphic to either*

$$\mathbf{P}_{\mathbf{P}^4}(\mathcal{O}_{\mathbf{P}^4} \oplus \mathcal{O}_{\mathbf{P}^4}(1)) \ \text{or}\ \mathbf{P}_{\mathbf{P}^4}(\mathcal{O}_{\mathbf{P}^4} \oplus \mathcal{O}_{\mathbf{P}^4}(3)).$$

Proof. In this case, X has a divisorial contraction whose image of the exceptional divisor is a point. So, all we have to do is to check the classified list in [Bo]. □

Proposition 3.2. *Let $X = X_\Sigma$ be a smooth toric Fano 5-fold of index 2. If X has an extremal primitive relation of type (4), then X is isomorphic to either*

$$\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3 \ \text{or}\ \mathbf{P}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1)).$$

Proof. In this case, X is a $\mathbf{P}^3$-bundle over a toric del Pezzo surface. By checking the classification of toric del Pezzo surfaces, we can prove this proposition. □

Proposition 3.3. *Let $X = X_\Sigma$ be a smooth toric Fano 5-fold of index 2. If X has an extremal primitive relation of type (5), then X is isomorphic to*

$$\mathbf{P}^1 \times \mathbf{P}_{\mathbf{P}^3}(\mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(2)).$$

Proof. In this case, X has a divisorial contraction whose image of the exceptional divisor is a curve. So, all we have to do is to check the classified list in [S3]. □

Proposition 3.4. *Let $X = X_\Sigma$ be a smooth toric Fano 5-fold of index 2. If X has an extremal primitive relation of type (8), then X has a $\mathbf{P}^1$-bundle structure. In this case, there exist exactly 9 such smooth toric Fano 5-folds of index 2 (see* Theorem 3.6*).*

Proof. By the classified list of smooth toric Fano 4-folds (see [Ba2, S2]), we have exactly 8 smooth toric Fano 4-folds whose indices are at least 2. It is an easy exercise to construct toric $\mathbf{P}^1$-bundles which are smooth toric Fano 5-folds of index 2 over them. □

Thus, we may assume that every extremal primitive relation of X is of type (6) or (7). However, there is no such variety as follows:

Lemma 3.5. *Let $X = X_\Sigma$ be a smooth toric Fano 5-fold of index 2. Then, X has a primitive relation of type other than (6) and (7).*

Proof. Suppose that every extremal primitive relation of X is of type (6) or (7). If the Picard number $\rho(X)$ of X is 2, then X has a Fano contraction (see [Kl]). So, we may assume $\rho(X) \geq 3$.

First of all, we claim that there is no primitive collection $P \in \mathrm{PC}(\Sigma)$ such that $\#P = 2$. This is obvious because P has to be an extremal primitive collection by Proposition 2.5. Namely, its primitive relation is of type (8).

Suppose that there exists an extremal primitive relation $x_1+x_2+x_3+x_4 = y_1+y_2$. Since $\rho(X) \geq 3$, there exist two distinct elements $z_1,\ z_2 \in \mathrm{G}(\Sigma) \setminus \{x_1,x_2,x_3,x_4,y_1,y_2\}$. $\{y_1,z_1\}$, $\{y_2,z_1\}$, $\{y_1,z_2\}$ and $\{y_2,z_2\}$ are not primitive collections. Thus, we have two extremal primitive relations

$$y_1+y_2+z_1 = w_1 \text{ and } y_1+y_2+z_2 = w_2,$$

where $w_1,\ w_2 \in \mathrm{G}(\Sigma)$. However, Proposition 2.4 says that $\{z_1,w_2\}$ and $\{z_2,w_1\}$ are primitive collections, and this is a contradiction.

Finally, we may assume that every extremal primitive relation of X is of type (7).

As in the argument of the case where there exists an extremal primitive relation of type (6), for any distinct extremal primitive collections $P_1,\ P_2 \in \mathrm{PC}(\Sigma)$ such that $\#P_1 = \#P_2 = 3$, we have $\#(P_1 \cap P_2) \neq 2$. So, let $\#(P_1 \cap P_2) = 1$, and let $x_1+x_2+x_3 = y_1$ and $x_1+x_4+x_5 = y_2$ be the corresponding primitive relations. Then, Propositon 2.4 imply that $\{x_2,x_3,y_2\}$ is a primitive collection. This primitive collection is extremal. So, the corresponding primitive relation is $x_2+x_3+y_2 = z$ for some $z \in \mathrm{G}(\Sigma)$. By applying Proposition 2.4 again, $\{x_1,z\}$ is a primitive collection. This is a contradiction. Therefore, $P_1 \cap P_2 = \emptyset$.

Since the Picard number of X is at least 3, there exist at least three extremal primitive collections P_1, P_2 and P_3. Thus, $\#\,\mathrm{G}(\Sigma) \geq 9$ and the Picard number of X is at least 4. So, we have a new extremal primitive collection P_4 and $\#\,\mathrm{G}(\Sigma) \geq 12$. We can continue this process endlessly. This is impossible. □

By Propositions 3.1, 3.2, 3.3, 3.4 and Lemma 3.5, we complete the classification:

Theorem 3.6. *Let $X = X_\Sigma$ be a smooth toric Fano 5-folds of index 2. Then, X is one of the follwing*:

(1) $\mathbf{P}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$.

(2) $\mathbf{P}_{\mathbf{P}^4}(\mathcal{O}_{\mathbf{P}^4} \oplus \mathcal{O}_{\mathbf{P}^4}(1))$.

(3) $\mathbf{P}_{\mathbf{P}^4}(\mathcal{O}_{\mathbf{P}^4} \oplus \mathcal{O}_{\mathbf{P}^4}(3))$.

(4) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3$.

(5) $\mathbf{P}^1 \times \mathbf{P}_{\mathbf{P}^3}(\mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(2))$.

(6) $\mathbf{P}^1$-bundle over $\mathbf{P}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$ whose primitive relations are $x_1+x_2+x_3 = x_4$, $x_4+x_5+x_6 = x_7$ and $x_7+x_8 = 0$, where $\mathrm{G}(\Sigma) = \{x_1,\ldots,x_8\}$.

(7) $\mathbf{P}^1$-bundle over $\mathbf{P}^2 \times \mathbf{P}^2$ whose primitive relations are $x_1+x_2+x_3 = x_7$, $x_4+x_5+x_6 = x_7$ and $x_7+x_8 = 0$, where $\mathrm{G}(\Sigma) = \{x_1,\ldots,x_8\}$.

(8) $\mathbf{P}^1$-bundle over $\mathbf{P}^2 \times \mathbf{P}^2$ whose primitive relations are $x_1+x_2+x_3 = x_7$, $x_4+x_5+x_6 = x_8$ and $x_7+x_8 = 0$, where $\mathrm{G}(\Sigma) = \{x_1,\ldots,x_8\}$.

(9) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$.

(10) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

Remark 3.7. There is another paper [BCDD] in which toric Fano manifolds of higher indices are studied. We remark that there is no explicit classified list of smooth toric Fano 5-folds of index 2 in [BCDD].

Remark 3.8. As in Theorem 3.6, the fan of every smooth toric Fano 5-fold of index 2 is a splitting fan. Moreover, if $d \leq 5$ and $p \geq 2$, then the fan of every smooth toric Fano d-fold of index p is a splitting fan. In section 4., we show higher dimensional examples of toric Fano manifolds of higher indicies which admit no projective space bundle structure.

4. Examples. In this section, we give examples of toric Fano manifolds of index 2 which admit no projective space bundle structure.

Let $X = X_\Sigma$ be a smooth complete toric d-fold. For any $x \in \mathrm{G}(\Sigma)$ and $p \in \mathbf{Z}_{\geq 2}$, we construct a new toric manifold $\mathcal{H}_{(x,p)}(X)$ as follows:

Put $\overline{N} := N \oplus \mathbf{Z}^{p-1}$ and let $\{e_1, \dots, e_d, e_{d+1}, \dots, e_{d+p-1}\}$ be the standard basis for $\overline{N}$. Put $z_1 := e_{d+1}, \dots, z_{p-1} := e_{d+p-1}$ and $z_p := x - (z_1 + \cdots + z_{p-1})$. We define a fan $\overline{\Sigma}$ in $\overline{N}$ as follows: The maximal cones of $\overline{\Sigma}$ are $\sigma + \mathbf{R}_{\geq 0} z_{i_1} + \cdots + \mathbf{R}_{\geq 0} z_{i_{p-1}}$, where σ is any maximal cone in Σ and $1 \leq i_1 < \cdots < i_{p-1} \leq p$. Then, obviously, we have an extremal primitive relation $z_1 + \cdots + z_p = x$ of $X_{\overline{\Sigma}}$. So, we obtain a smooth complete toric $(d + p - 1)$-fold $\mathcal{H}_{(x,p)}(X)$ by the corresponding blow-down. It is obvious that the Picard number of $\mathcal{H}_{(x,p)}(X)$ is same as X. Moreover, $\mathcal{H}_{(x,p)}(X)$ has the following property:

Proposition 4.1. *The primitive collections of* $\mathcal{H}_{(x,p)}(X)$ *are*

(1) $P \in \mathrm{PC}(\Sigma)$, where $x \notin P$, and

(2) $(P \setminus \{x\}) \cup \{z_1, \dots, z_p\}$, where $P \in \mathrm{PC}(\Sigma)$ and $x \in P$.

Moreover, if $x + x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n$ *is a primitive relation of* X*, then* $z_1 + \cdots + z_p + x_1 + \cdots + x_m = b_1 y_1 + \cdots + b_n y_n$ *is a primitive relation of* $\mathcal{H}_{(x,p)}(X)$*, while if* $x_1 + \cdots + x_m = bx + b_1 y_1 + \cdots + b_n y_n$ *is a primitive relation of* X*, then* $x_1 + \cdots + x_m = bz_1 + \cdots + bz_p + b_1 y_1 + \cdots + b_n y_n$ *is a primitive relation of* $\mathcal{H}_{(x,p)}(X)$.

Proof. The primitive collections of $\overline{\Sigma}$ are the primitive collections of Σ and $\{z_1, \dots, z_p\}$. Then, we can calculate the primitive collections of $\mathcal{H}_{(x,p)}(X)$ easily (see [S1, Corollary 4.9]). □

Now, we can describe an example of a toric Fano manifold of index 2 which admits no projective space bundle structure.

Example 4.2. Let $X = X_\Sigma$ be the del Pezzo surface of degree 7. The primitive relations of Σ are $x_1 + x_3 = x_2$, $x_1 + x_4 = 0$, $x_2 + x_4 = x_3$, $x_2 + x_5 = x_1$ and $x_3 + x_5 = 0$, where $\mathrm{G}(\Sigma) = \{x_1, x_2, x_3, x_4, x_5\}$. Put

$$Y = Y_{\widetilde{\Sigma}} := \mathcal{H}_{(x_1,2)}\left(\mathcal{H}_{(x_2,2)}\left(\mathcal{H}_{(x_3,2)}\left(\mathcal{H}_{(x_4,2)}\left(\mathcal{H}_{(x_5,2)}(X)\right)\right)\right)\right).$$

Then, the primitive relations of $\widetilde{\Sigma}$ are

$$x_1 + x_1' + x_3 + x_3' = x_2 + x_2',\ x_1 + x_1' + x_4 + x_4' = 0,$$

$$x_2 + x_2' + x_4 + x_4' = x_3 + x_3',\ x_2 + x_2' + x_5 + x_5' = x_1 + x_1' \text{ and}$$

$$x_3 + x_3' + x_5 + x_5' = 0,$$

where $\mathrm{G}(\widetilde{\Sigma}) = \{x_1, x_2, x_3, x_4, x_5, x_1', x_2', x_3', x_4', x_5'\}$. We remark that Y is a smooth toric Fano 7-fold of index 2, the Picard number of Y is 3 and Y has no projective space bundle structure.

Remark 4.3. Similarly as in Example 4.2, for any $p \in \mathbf{Z}_{\geq 2}$, we can construct a toric Fano manifold of index p which has no projective space bundle structure.

The following is a 6-dimensional example of a toric Fano manifold of index 2 which has no projective space bundle structure. This Fano manifold can not be constructed from a lower-dimensional Fano manifold as in Example 4.2.

Example 4.4. Let $X = X_\Sigma$ be a smooth toric Fano 6-fold whose primitive relations are $x_1 + x_2 + x_3 + x_4 = 0$, $x_5 + x_6 + x_7 + x_8 = 0$, $x_4 + x_7 + x_8 = x_9$, $x_5 + x_6 + x_9 = x_4$ and $x_1 + x_2 + x_3 + x_9 = x_7 + x_8$, where $\mathrm{G}(\Sigma) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. This is the simplest example of a toric Fano manifold of index 2 which has no projective space bundle structure. For a line $L \subset \mathbf{P}^3$ and a plane $P \subset \mathbf{P}^3$, X is obtained by the blow-up of $\mathbf{P}^3 \times \mathbf{P}^3$ along $L \times P$.

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