Harmonic Teichmüller mappings

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Abstract: In this paper we present a necessary and sufficient condition for a C^2 Teichmüller mapping to be ρ-harmonic. By this result we show that there is no solution to the Schoen conjecture in the class of C^2 Teichmüller mappings. We also obtain two characterizations for π-harmonic Teichmüller mappings.

Key words: Quasiconformal mappings; Teichmüller mappings; harmonic mappings.

1. Introduction. Let Ω and Ω' be two Jordan domains of the complex plane $\mathbf{C}$. $\rho(w)|dw|^2$ is a *singular metric* of Ω' if its metric density ρ is a nonnegative smooth function and admits isolated zeros on Ω'. Given a holomorphic quadratic differential $\psi(w)dw^2$ on Ω', $|\psi(w)||dw|^2$ defines a *singular flat metric* with singularities at the zeros of ψ. In this paper, a C^2 mapping $f:\Omega\to\Omega'$ is said to be *harmonic with respect to* ρ (shortly *ρ-harmonic*) if it satisfies

$$\partial_{\bar z}(\rho(f)f_z\bar f_z)=0, \tag{1.1}$$

that is, the *Hopf differential* $\rho(f)f_z\bar f_z dz^2$ of f is holomorphic in Ω. Gromov and Schoen [4] first systematically studied harmonic mappings with respect to a singular metric in connection with arithmetic superrigidity.

If a ρ-harmonic mapping f from Ω onto Ω' is also quasiconformal, then we call it a *ρ-harmonic quasiconformal mapping* (abbreviatedly *ρ-HQC mapping*). Particularly, we say that f is a *π-HQC mapping* if ρ is a constant. It is well known that the Jacobian J_f of each ρ-HQC mapping f is always positive since f is a C^2 quasiconformal mapping. The Beltrami coefficient μ_f of a ρ-HQC mapping f is of the form

$$\mu_f=|\mu_f|\frac{\overline{\varphi}}{|\varphi|},$$

where $\varphi=\rho(f)f_z\bar f_z$ (see [16]).

A ρ-HQC mapping f is a Teichmüller mapping if $|\mu_f|$ is a constant k with $0<k<1$, then we call f a *ρ-harmonic Teichmüller mapping* and $\varphi(z)dz^2$ the associated holomorphic quadratic differential of f. In particular, f is said to be a *π-harmonic Teichmüller mapping* if ρ is a constant. Let $\phi(w)dw^2$ be the associated holomorphic quadratic differential of f^{-1}, then f induces a singular flat metric $|\phi(w)||dw|^2$ on Ω'. One can refer to [5, 6] for some results about harmonic Teichmüller mappings with respect to singular flat metrics.

Beurling and Ahlfors [2] proved that a homeomorphism from the unit circle onto itself admits a quasiconformal extension from the unit disk onto itself if and only if it is quasisymmetric. In order to prove that there always exists an extension with good properties, Schoen [19] posed the following conjecture.

Schoen conjecture. The harmonic quasiconformal homeomorphisms with respect to the Poincaré metric from the unit disk onto itself are parameterized by the set of quasisymmetric homeomorphisms of the unit circle onto itself.

There are some partial results on the above conjecture. When the boundary mapping is also sufficiently differentiable, Li and Tam [7] proved that there exists a HQC mapping with respect to the Poincaré metric. Later, Li [8] and Markovic [11] generalized their result. Since Teichmüller mappings play a vital role in the theory of quasiconformal mappings, the following question is natural.

Question 1. Does there exist a solution to the Schoen conjecture in the class of C^2 Teichmüller mappings?

It is equivalent to ask whether a Teichüller mapping can be a HQC mapping with respect to the Poincaré metric. Namely, does there exist a HQC

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mapping with respect to the Poincaré metric such that the modulus of its Beltrami coefficient is a constant? In order to deal with it, we only need to answer the following question.

Question 2. For a C^2 Teichmüller mapping f, does there uniquely exist a metric $\rho(w)|dw|^2$ such that f is ρ-harmonic?

In Section 2, we will answer Question 2 (see Theorem 2.1). In fact we will prove that a C^2 Teichmüller mapping f is harmonic with respect to a singular flat metric, which is determined by the associated holomorphic quadratic differential of f^{-1}. Then we show that such metric is unique ignoring multiplying a positive constant. As an application of Theorem 2.1 to the Schoen conjecture, we will give a negative answer to Question 1 in Section 3.

Martio [12] first studied the existence of π-HQC mappings. Recently, there is much progress in prescribing the properties of π-HQC mappings in [3, 9, 10, 13–15]. As another application of Theorem 2.1, we will prescribe some properties of a π-harmonic Teichmüller mapping in Section 4. It is well known that a univalent π-harmonic mapping f has a canonical representation $f = h + \bar{g}$ where g and h are holomorphic. Furthermore, $g = \bar{\alpha} h$ if f is also a Teichmüller mapping, where h is a conformal mapping and α is a constant such that $0 < |\alpha| < 1$.

2. A necessary and sufficient condition for a C^2 Teichmüller mapping to be ρ-harmonic. In this section, we will prove the existence for singular metrics by Lemma 2.1, and then obtain a sufficient condition for a C^2 Teichmüller mapping to be ρ-harmonic. By Lemma 2.2, we show that the condition is also necessary.

Lemma A. ([17, 18]) *If f is a Teichmüller mapping from the unit disk Δ onto itself then its inverse mapping $g = f^{-1}$ is also a Teichmüller mapping.*

Remark. For a Teichmüller mapping from Ω onto Ω' the result of Lemma A also holds.

Lemma 2.1. *If f is a C^2 Teichmüller mapping from Ω onto Ω' and the associated holomorphic quadratic differential of its inverse mapping is $\phi(w)dw^2$ then f is $c|\phi|$-harmonic, where c is a positive constant.*

Proof. Denote by F the inverse mapping of f. We know that F is also a Teichmüller mapping by Lemma A. Let $\phi(w)dw^2$ be the associated holomorphic quadratic differential of F. Set $\psi^2(w) = \phi(w)$. Then there exists a constant k, $0 < k < 1$, such that

$$\mu_F = \frac{F_{\bar{w}}}{F_w} = k\frac{\overline{\phi}}{|\phi|} = k\frac{\overline{\psi^2}}{|\psi^2|} = k\frac{\overline{\psi}}{\psi}. \tag{2.1}$$

By differentiating both sides of (2.1) in w, we have

$$F_{ww} = \frac{F_{\bar{w}w}\psi}{k\overline{\psi}} + \frac{F_{\bar{w}}\psi'}{k\overline{\psi}}. \tag{2.2}$$

Hence

$$\begin{aligned}(c|\phi(f)|f_z\overline{f_{\bar{z}}})_{\bar{z}} &= -\frac{ck^2}{(1-k^2)^2}(\frac{|\phi|}{F_wF_{\bar{w}}}\circ f)_{\bar{z}}\\ &= -\frac{ck}{(1-k^2)^2}(\frac{\phi}{F_w^2}\circ f)_{\bar{z}}.\end{aligned}$$

Since $F_w \circ f = \overline{f_z}/J_f$ and $-F_{\bar{w}} \circ f = f_{\bar{z}}/J_f$ (see [1, p. 8]), we have

$$\begin{aligned}(c|\phi(f)|f_z\overline{f_{\bar{z}}})_{\bar{z}} &= \frac{2ck}{(1-k^2)^2}\cdot\\ &(\frac{\psi\psi' F_w^2F_{\bar{w}} - \psi^2F_wF_{\bar{w}}F_{ww} + \psi^2F_w^2F_{w\bar{w}}}{J_FF_w^4}\circ f).\end{aligned} \tag{2.3}$$

So it follows from (2.1) and (2.2) that

$$(c|\phi(f)|f_z\overline{f_{\bar{z}}})_{\bar{z}} = 0.$$

Therefore f is a $c|\phi|$-harmonic mapping, and the proof is completed. □

Next we will prove that the metric is unique ignoring multiplying a positive constant.

Lemma 2.2. *If f is a ρ-harmonic Teichmüller mapping from Ω onto Ω', then*

$$\rho(w) = c|\phi(w)| \quad (w \in \Omega'), \tag{2.4}$$

where $\phi(w)dw^2$ is the associated holomorphic quadratic differential of f^{-1}, and c is a positive constant.

Proof. By the hypothesis we have

$$(\rho(f)f_z\overline{f_{\bar{z}}})_{\bar{z}} = 0 \ (z \in \Omega). \tag{2.5}$$

Let F be the inverse mapping of f. We assume that J_f and J_F are the Jacobians of f and F, respectively. Since $F_w \circ f = \overline{f_z}/J_f$ and $-F_{\bar{w}} \circ f = f_{\bar{z}}/J_f$, the relation (2.5) is equivalent to

$$\left(\frac{\rho(w)F_wF_{\bar{w}}}{J_F^2}\circ f\right)_z = 0 \quad (w = f(z), \quad z \in \Omega). \tag{2.6}$$

By the chain rule we have

$$(2.7)\qquad \Big(\frac{\rho F_w F_{\bar w}}{J_F^2}\Big)_{\bar w}\cdot\Big(-\frac{\overline{F_{\bar w}}}{J_F}\Big)+ \Big(\frac{\rho F_w F_{\bar w}}{J_F^2}\Big)_{w}\cdot\Big(\frac{\overline{F_w}}{J_F}\Big)=0 \quad (w\in\Omega').$$

Lemma A implies that F is also a Teichmüller mapping from Ω' onto Ω. Set $\phi(w)=\psi^2(w)$, then the Beltrami coefficient of F is of the form

$$(2.8)\qquad \mu_F=\frac{F_{\bar w}}{F_w}=k\frac{\overline{\phi}}{|\phi|}=k\frac{\overline{\psi^2}}{|\psi^2|}=k\frac{\overline{\psi}}{\psi}.$$

By the relation $\dfrac{k^2}{\overline{\mu_F}}=\mu_F$, we reduce (2.7) to

$$\mu_F\Big(\frac{\rho}{F_wF_{\bar w}}\Big)_w=\Big(\frac{\rho}{F_wF_{\bar w}}\Big)_{\bar w}.$$

Hence

$$\mu_F\frac{\rho_wF_wF_{\bar w}-\rho(F_{ww}F_{\bar w}+F_wF_{\bar w w})}{(F_wF_{\bar w})^2}= \frac{\rho_{\bar w}F_wF_{\bar w}-\rho(F_{w\bar w}F_{\bar w}+F_wF_{\bar w\bar w})}{(F_wF_{\bar w})^2}.$$

By direct calculations it follows that

$$(2.9)\qquad (\mu_F\rho_w-\rho_{\bar w})F_wF_{\bar w} + \rho(F_wF_{\bar w\bar w}-\mu_FF_{\bar w}F_{ww})=0.$$

From (2.8) we have

$$(2.10)\qquad F_{ww}=\frac{F_{\bar w w}\psi}{k\overline{\psi}}+\frac{F_{\bar w}\psi'}{k\overline{\psi}}$$

and

$$(2.11)\qquad F_{\bar w\bar w}=\frac{kF_{\bar w w}\overline{\psi}}{\psi}+\frac{kF_w\overline{\psi'}}{\psi}.$$

Thus by (2.10) and (2.11), the relation (2.9) can be simplified as

$$\frac{\mu_F\rho_w-\rho_{\bar w}}{\rho}=\mu_F\frac{\psi'}{\psi}-\frac{\overline{\psi'}}{\overline{\psi}}.$$

Since ρ is a real function we obtain

$$(2.12)\qquad \mu_F\Big(\log\frac{\rho}{\psi}\Big)_w=\overline{\Big(\log\frac{\rho}{\psi}\Big)_w}.$$

Noting that $0<|\mu_F|=k<1$, we have

$$\Big(\log\frac{\rho}{\psi}\Big)_w=0,$$

that is,

$$\Big(\log\frac{\rho}{\overline{\psi}}\Big)_{\bar w}=0.$$

So $\rho/\overline{\psi}$ is analytic, namely, $\dfrac{\rho}{|\psi|^2}\psi$ is analytic. Therefore $\rho/|\psi|^2$ (*i. e.*, $\rho/|\phi|$) is a positive constant. $\square$

Combining Lemma 2.1 with Lemma 2.2, we obtain

Theorem 2.1. *If f is a C^2 Teichmüller mapping from Ω onto Ω' and the associated holomorphic quadratic differential of its inverse mapping is $\phi(w)dw^2$ then f is a ρ-harmonic mapping if and only if $\rho=c|\phi|$, where c is a positive constant.*

3. ρ_P-harmonic Teichmüller mappings.

In this section we will give a negative answer to Question 1. In fact we will obtain a more general result (see Theorem 3.1). The Poincaré metric $\rho_P|dz|^2$ of the unit disk of Δ is $|dz|^2/(1-|z|^2)^2$. Generally, the Poincaré metric $\rho_P|dw|^2$ of Ω is given by $|\phi'(w)|^2|dw|^2/(1-|\phi(w)|^2)^2$, where ϕ is a conformal mapping from Ω onto Δ. Let Ω_i $(i=0,1,2,3)$ be Jordan domains of the complex plane $\mathbf{C}$.

Lemma 3.1. *Let ϕ be a conformal mapping from Ω_2 onto Ω_3 and φ be a conformal mapping from Ω_0 onto Ω_1. If $F:\Omega_1\to\Omega_2$ is harmonic with respect to ρ_2, then $\phi\circ F$ is harmonic with respect to ρ_3, and $F\circ\varphi$ is harmonic with respect to ρ_2, where ρ_i is the Poincaré metric density of Ω_i (i=2,3), respectively.*

Proof. If F is harmonic with respect to ρ_2, then by definition $\rho_2(F(z))F_z\overline{F_{\bar z}}$ is holomorphic in Ω_1. If ϕ is a conformal mapping from Ω_2 onto Ω_3, then $\phi\circ F$ is a C^2 homeomorphism from Ω_1 onto Ω_3 and

$$\begin{aligned}\rho_3(\phi\circ F)(\phi\circ F)_z\overline{(\phi\circ F)_{\bar z}}&=\rho_3(\phi\circ F)|\phi'(w)|^2F_z\overline{F_{\bar z}}\\ &=\rho_3(\phi(w))|\phi'(w)|^2F_z\overline{F_{\bar z}}=\frac{\rho_2(w)}{|\phi'(w)|^2}|\phi'(w)|^2F_z\overline{F_{\bar z}}\\ &=\rho_2(F(z))F_z\overline{F_{\bar z}},\end{aligned}$$

where $w=F(z)$. Thus, $\rho_3(\phi\circ F)(\phi\circ F)_z\overline{(\phi\circ F)_{\bar z}}$ is holomorphic in Ω_1, that is, $\phi\circ F$ is harmonic with respect to ρ_3.

Let φ be a conformal mapping from Ω_0 onto Ω_1. We therefore obtain that $F\circ\varphi$ is a C^2 homeomorphism of Ω_0 onto Ω_2 and

$$\begin{aligned}&\rho_2(F\circ\varphi(\zeta))(F\circ\varphi)_\zeta\overline{(F\circ\varphi)_{\bar\zeta}}\\ &=\rho_2(F\circ\varphi(\zeta))F_z(\varphi(\zeta))\varphi'(\zeta)\overline{F_{\bar z}(\varphi(\zeta))}\overline{\varphi'(\zeta)}\\ &=\rho_2(F(z))F_z(z)\overline{F_{\bar z}(z)}(\varphi'(\zeta))^2,\end{aligned}$$

where $z = \varphi(\zeta)$. Since $\rho_2(F)F_z\overline{F_{\bar{z}}}$ is holomorphic, we can write $\psi(z) = \rho_2(F)F_z\overline{F_{\bar{z}}}$. So $\psi(\varphi(\zeta))$ is holomorphic in Ω_0. Thus $\psi(\varphi(\zeta))(\varphi'(\zeta))^2$ is holomorphic in Ω_0, that is, $\rho_2(F \circ \varphi(\zeta))(F \circ \varphi)_\zeta\overline{(F \circ \varphi)_{\bar{\zeta}}}$ is holomorphic in Ω_0. Hence $F \circ \varphi$ is harmonic with respect to ρ_2. □

Theorem 3.1. *There does not exist any ρ_P-harmonic Teichmüller mapping from Ω onto Ω', where ρ_P is the Poincaré metric density of Ω'.*

Proof. By Lemma 3.1, we can choose Ω' to be the unit disk Δ. Suppose that there exists a ρ_P-harmonic Teichmüller mapping of Ω onto Δ. Then from Theorem 2.1, there exists a holomorphic function $\phi(w)$, $w \in \Delta$ such that

$$1/(1-|w|^2)^2 = c|\phi(w)|, \tag{3.1}$$

where c is a positive constant.

Write $\psi^2(w) = \phi(w)$. Differentiating both sides of (3.1) in w, we obtain

$$c\overline{\psi'} = 2\frac{w}{\psi}\frac{1}{(1-|w|^2)^3}. \tag{3.2}$$

By differentiating in w again, we have

$$\frac{2}{\psi(1-|w|^2)^3}(2+\frac{\psi'}{\psi}w - \frac{3}{1-|w|^2}) = 0.$$

Thus

$$\frac{\psi'}{\psi}w = \frac{3}{1-|w|^2} - 2. \tag{3.3}$$

The left side of (3.3) is a meremorphic function but the right side of (3.3) is a real function and is not a constant, a contradiction. The proof is completed. □

4. π-harmonic Teichmüller mappings. In this section we will give another application of Theorem 2.1 to present some characterizations of π-harmonic Teichmüller mappings.

Theorem 4.1. *If f is a C^2 Teichmüller mapping from Ω onto Ω' then the following conditions are equivalent*

(i) f is π-harmonic;

(ii) the Beltrami coefficient of its inverse mapping is a constant;

(iii) f is of the form $f = h + \alpha\bar{h}$, where α is a constant with $0 < |\alpha| < 1$ and h is a conformal mapping in Ω.

Proof. It is obvious that *(iii)* implies *(i)*. Suppose that f is a C^2 Teichmüller mapping from Ω onto Ω'. Let F be the inverse of f and $\phi(w)dw^2$ the associated holomorphic quadratic differential of F. It follows from Theorem 2.1 that f is π-harmonic if and only if ϕ is a constant. Thus we see that *(i)* is equivalent to *(ii)*.

Let μ_F be the Beltrami coefficient of the inverse mapping F. Suppose that μ_F is a constant $-b$ with $|b| = k$. Using the formulas $F_w \circ f = \overline{f_z}/J_f$ and $-F_{\bar{w}} \circ f = f_{\bar{z}}/J_f$ again, we have

$$-\frac{f_{\bar{z}}}{\overline{f_z}} = \mu_F \circ f = -b.$$

Since f is a univalent π-harmonic mapping, there exist two holomorphic functions m and n such that $f = m + \bar{n}$. Thus

$$n' = \bar{b}m'. \tag{4.1}$$

So $n = \bar{b}m + d$, where d is a constant. Therefore $f = m + \overline{\bar{b}m + d}$. Set $h = m + (\bar{d} - bd)/(1-|b|^2)$. Then there exists a holomorphic function h such that $f = h + b\bar{h}$, where $|b| = k$ and $0 < k < 1$.

For two arbitrary points z_1 and z_2 in Ω, we get

$$f(z_1) - f(z_2) = h(z_1) - h(z_2) + b(\overline{h(z_1)} - \overline{h(z_2)}).$$

Hence f is univalent if and only if h is conformal in Ω. Therefore there exists a conformal mapping h in Ω such that $f = h + b\bar{h}$, where $|b| = k$ and $0 < k < 1$. Thus *(ii)* implies *(iii)*. □

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