On the ranks of Conway group Co_1

By Faryad Ali and Mohammed Ali Faya Ibrahim

Department of Mathematics, King Khalid University,
P.O. Box 9004, Abha, Saudi Arabia

(Communicated by Shigefumi Mori, M. J. A., June 14, 2005)

Abstract: Let G be a finite group and X a conjugacy class of G. We denote rank$(G : X)$ to be the minimum number of elements of X generating G. In the present paper we investigate the ranks of the Conway group Co_1. Computations were carried with the aid of computer algebra system **GAP** [16].

Key words: Conway's group Co_1; rank; generator; sporadic group.

1. Introduction and preliminaries. Let G be a finite group and $X \subseteq G$. We denote the minimum number of elements of X generating G by rank$(G : X)$. In the present paper we investigate rank$(G : X)$ where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [12, 13] and [14] proved that rank$(Fi_{22} : 2A) \in \{5, 6\}$ and rank$(Fi_{22} : 2B) =$ rank$(Fi_{22} : 2C) = 3$ where $2A$, $2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as represented in the **ATLAS** [4]. The work of Hall and Soicher [10] shows that rank$(Fi_{22} : 2A) = 6$. Moori in [15] determined the ranks of the Janko group J_1, J_2 and J_3. Recently in [1] and [2] the authors computed the ranks of the four sporadic simple groups HS, McL, Co_2 and Co_3.

In the present article, the authors continue their study to determine the ranks of the sporadic simple groups and the problem is resolved for the Conway's largest sporadic simple group Co_1. We determine the rank for each conjugacy class of Co_1. We prove the following result:

Theorem 2.7. *Let Co_1 be the Conway's largest sporadic simple group. Then*
(a) rank$(Co_1 : nX) = 3$ *if* $nX \in \{2A, 2B, 2C, 3A\}$.
(b) rank$(Co_1 : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 2C, 3A\}$.

For basic properties of Co_1, character tables of Co_1 and their maximal subgroups we use **ATLAS** [4] and **GAP** [16]. For detailed information about the computational techniques used in this paper the reader is encouraged to consult [1, 9] and [14].

Throughout this paper our notation is standard and taken mainly from [1, 2] and [9]. In particular, for a finite group G with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can be easily computed from the character table of G (see [11], p. 45) by the following formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2|\cdots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G. Further let $\Delta^*_G(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \cdots g_{k-1} = g_k$ such that $G = \langle g_1, g_2, \ldots, g_{k-1}\rangle$. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then we say that G is $(C_1, C_2, \ldots, C_k)$-generated. If H is a subgroup of G containing g_k and B is a conjugacy class of H such that $g_k \in B$, then $\Sigma_H(C_1, C_2, \ldots C_{k-1}, B)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_i \in C_i$ and $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1}\rangle \leq H$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to **ATLAS** [4]. A general conjugacy class of elements of order n in G is denoted by nX. For example $2A$ represents the first conjugacy class of invo-

2000 Mathematics Subject Classification. Primary 20D08, 20F05.

Dedicated to Prof. Jamshid Moori on the occasion of his sixtieth birthday.

lutions in a group G. We will use the maximal subgroups and the permutation characters of Co_1 on the conjugates (right cosets) of the maximal subgroups listed in the **ATLAS** [4] extensively.

The following results will be crucial in determining the ranks of a finite group G.

Lemma 1.1 (Moori [15]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \ldots, lX}_{m\text{-times}}, (nZ)^m)$-generated.*

Corollary 1.2. *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then* $\mathrm{rank}(G : lX) \leq m$.

Proof. The proof follows immediately from Lemma 1.1. □

Lemma 1.3 (Conder *et al.* [5]). *Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

We will employ results that, in certain situations, will effectively establish non-generation. They include Scott's theorem (*cf.* [5] and [17]) and Lemma 3.3 in [19] which we state here.

Lemma 1.4 ([19]). *Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

2. Ranks of Co_1.

The Conway group Co_1 is a sporadic simple group of order

$$4,157,776,806,543,360,000 = 2^{21}.3^9.5^4.11.13.23.$$

The subgroup structure of Co_1 is discussed in Wilson [18]. The group Co_1 has exactly 22 conjugacy classes of maximal subgroups as listed in Wilson [18]. Co_1 has 101 conjugacy classes of its elements. It has precisely three classes of involutions, namely $2A$, $2B$ and $2C$ as represented in the **ATLAS** [4]. Co_1 acts on a 24-dimensional vector space Ω over $GF(2)$ and this action produces three orbits on the set of non-zero vectors. The point stabilizers are the groups Co_2, Co_3 and $2^{11} : M_{24}$ and the permutation character of Co_1 on $\Omega - \{0\}$, which is given in [6], is $\chi = 3.1a + 2.299a + 2.17250a + 3.80730a + 376740a + 644644a + 2055625a + 2417415a + 2.5494125a$, where na denotes the first irreducible character with degree n. For basic properties of Co_1 and information on its maximal subgroups the reader is referred to [3, 4, 6] and [18].

Recently Darafsheh, Ashrafi and Moghani in [6, 7] and [8] established (p, q, r)-generations and nX-complementary generations of the Conway group Co_1. We will make use of these generations to determine the ranks of Co_1 in some cases.

In the following we prove that the Conway group Co_1 can be generated by three involutions.

Lemma 2.1. *The group Co_1 can be generated by three involutions $a, b, c \in 2A$ such that $abc \in 13A$.*

Proof. Using the character table of Co_1 we have $\Delta_{Co_1}(2A, 2A, 2A, 13A) = 9633$. In Co_1 we have only two maximal subgroups, up to isomorphism, with orders divisible by 13, namely, $H_1 \cong 3.Suz.2$ and $H_2 \cong (A_4 \times G_2(4)) : 2$. We also have

$$\Sigma_{H_1}(2A, 2A, 2A, 13A) = \Delta_{H_1}(2A, 2A, 2A, 13A) = 1521.$$

A fixed element of order 13 in Co_1 lies in four conjugates of H_1. Hence H_1 contributes $4 \times 1521 = 6084$ to the number $\Delta_{Co_1}(2A, 2A, 2A, 13A)$. Similarly, we compute that

$$\Sigma_{H_2}(2A, 2A, 2A, 13A) = \Delta_{H_2}(2A, 2A, 2A, 13A) = 169.$$

And a fixed element of order 13 in Co_1 lies in a unique conjugate of H_2. This means that H_2 contributes $1 \times 169 = 169$ to the number $\Delta_{Co_1}(2A, 2A, 2A, 13A)$. Since

$$\Delta^*_{Co_1}(2A, 2A, 2A, 13A) \geq 9633 - 6084 - 169 > 0,$$

the group Co_1 is $(2A, 2A, 2A, 13A)$-generated. □

Lemma 2.2. *Let Co_1 be the Conway's largest sporadic group Co_1 then* $\mathrm{rank}(Co_1 : 2X) = 3$ *where $X \in \{A, B, C\}$.*

Proof. We proved in the previous lemma that Co_1 is $(2A, 2A, 2A, 13A)$-generated and so $\mathrm{rank}(Co_1 : 2A) \leq 3$ but $\mathrm{rank}(Co_1 : 2A) = 2$ is not possible, because if $\langle x, y\rangle = Co_1$ for some $x, y \in 2A$ then $Co_1 \cong D_{2n}$ with $o(xy) = n$. Hence $\mathrm{rank}(Co_1 : 2A) = 3$. Darafsheh *et al.* in [6] proved that Co_1 is $(2Y, 3D, 11A)$-generated for $Y \in \{B, C\}$. Now by applying Corollary 1.2, we have $\mathrm{rank}(Co_1 : 2Y) \leq 3$ for $Y \in \{B, C\}$, but we know that $\mathrm{rank}(Co_1 : 2Y) > 2$ as we argue in the above case, hence $\mathrm{rank}(Co_1 : 2Y) = 3$ where $Y \in \{B, C\}$. Therefore $\mathrm{rank}(Co_1 : 2X) = 3$ where $X \in \{A, B, C\}$. □

Lemma 2.3. $\mathrm{rank}(Co_1 : 3A) = 3$.

Proof. First we show that $\mathrm{rank}(Co_1 : 3A) > 2$ by proving that Co_1 is not $(3A, 3A, tX)$-generated for any conjugacy class tX. If Co_1 is $(3A, 3A, tX)$-generated then $1/3 + 1/3 + 1/t < 1$ and it follows that $t \geq 4$. Set $K = \{4A, 5A, 6A\}$. Using **GAP** [16] we see that $\Delta_{Co_1}(3A, 3A, tX) = 0$ if $tX \notin K$ and for $tX \in K$ we have $\Delta_{Co_1}(3A, 3A, tX) < |C_{Co_1}(tX)|$. We get that

$$\Delta^*_{Co_1}(3A, 3A, tX) < \Delta_{Co_1}(3A, 3A, tX) < |C_{Co_1}(tX)|.$$

Using Lemma 1.4, we obtain that $\Delta^*_{Co_1}(3A, 3A, tX) = 0$ for all tX with $t \geq 4$ and therefore Co_1 is not $(3A, 3A, tX)$-generated and hence rank$(Co_1 : 3A) > 2$. Next we show that rank$(Co_1 : 3A) = 3$.

Consider the triple $(3A, 3A, 3A, 10E)$. From the maximal subgroups of Co_1, we see that the only maximal subgroups of Co_1 with order divisible by 10 and non-empty intersection with the conjugacy classes $3A$ and $10E$ are isomorphic to $H_1 = 2^{1+8}_+.O^+_8(2)$, $H_2 = 3^{1+4}.2U_4(2).2$, $H_3 = (A_5 \times J_2) : 2$ and $H_4 = (D_{10} \times (A_5 \times A_5).2).2$. We compute $\Delta_{Co_1}(3A, 3A, 3A, 10E) = 600$ and $\Sigma_{H_1}(3A, 3A, 3A, 10E) = \Sigma_{H_2}(3A, 3A, 3A, 10E) = \Sigma_{H_3}(3A, 3A, 3A, 10E) = \Sigma_{H_4}(3A, 3A, 3A, 10E) = 0$. Thus no proper subgroup of Co_1 is $(3A, 3A, 3A, 10E)$-generated and we get

$$\Delta^*_{Co_1}(3A, 3A, 3A, 10E) = \Delta_{Co_1}(3A, 3A, 3A, 10E) = 600.$$

Hence Co_1 is $(3A, 3A, 3A, 10E)$-generated and the result follows. □

Lemma 2.4. *rank$(Co_1 : tX) = 2$ for $tX \in \{3B, 4A, 4B, 4C, 4D, 5A, 6A\}$.*

Proof. Set $T = \{3B, 4B, 4D, 5A, 6A\}$. Consider the triple $(tX, tX, 13A)$ for each $tX \in T$. The maximal subgroups of Co_1 containing elements of order 13 are, up to isomorphism, $H_1 \cong 3.Suz.2$ and $H_2 \cong (A_4 \times G_2(4)) : 2$. We see that a fixed element of order 13 in Co_1 is contained in precisely four copies of H_1 in Co_1 and in a unique conjugate copy of H_2 in Co_1. Now we calculate that for each $tX \in T$, we have

$$\Delta^*_{Co_1}(tX, tX, 13A) \geq \Delta_{Co_1}(tX, tX, 13A) - 4\Sigma_{H_1}(tX, tX, 13A) - \Sigma_{H_2}(tX, tX, 13A) > 0.$$

We conclude that Co_1 is $(tX, tX, 13A)$-generated for each $tX \in T$. Hence rank$(Co_1 : tX) = 2$ for each $tX \in T$.

Next for $tX = 4A$ consider the triple $(2C, 4A, 26A)$. Up to isomorphism, the only maximal subgroup of Co_1 that may contain (2C,4A,26A)-generated proper subgroup is isomorphic to $H_2 \cong (A_4 \times G_2(4)) : 2$. We calculate that $\Delta_{Co_1}(2C, 4A, 26A) = 91$ and $\Sigma_{H_2}(2C, 4A, 26A) = 39$. Now a fixed element of order 26 in Co_1 lies in a unique conjugate of H_2 in Co_1. Hence H_2 contributes $1 \times 39 = 39$ to the number $\Delta_{Co_1}(2C, 4A, 26A)$. Our calculation gives $\Delta^*_{Co_1}(2C, 4A, 26A) \geq 91 - 39 > 0$ and therefore, Co_1 is $(2C, 4A, 26A)$-generated. Now applying Lemma 1.2, we get rank$(Co_1 : 4A) = 2$.

Finally for the rank of the conjugacy class $tX = 4C$ we consider the triple $(4C, 4C, 13A)$. The Co_1-class $4C$ fails to meet any copy of H_1 or H_2 in Co_1. Thus Co_1 contains no proper $(4C, 4C, 13A)$-subgroup. As $\Delta_{Co_1}(4C, 4C, 13A) = 7866268$ we conclude that Co_1 is $(4C, 4C, 13A)$-generated and rank$(Co_1 : 4C) = 2$. This completes the proof. □

Lemma 2.5. *If $n \geq 4$ and $nX \notin T = \{4A, 4B, 4C, 4D, 5A, 6A\}$ then* rank$(Co_1 : nX) = 2$.

Proof. Direct computation using **GAP** and results from Darafsheh, Ashrafi and Moghani ([8]) together with information about the power maps of Co_1 we can show that Co_1 is $(2A, nX, mZ)$-generated for each conjugacy class $nX \notin T$ of Co_1 $(n \geq 4)$ with appropriate mZ. Now by Lemma 1.3, Co_1 is $(nX, nX, (mZ)^2)$-generated for all $nX \notin T$ $(n \geq 4)$. Hence rank$(Co_1 : nX) = 2$ for all $n \geq 4$ and for each conjugacy class $nX \notin T$ of Co_1. □

Remark 2.6. For example Co_1 is $(2A, 23A, 23B)$-generated. Hence Co_1 is $(23A, 23A, (23B)^2)$-generated, so that rank$(Co_1 : 23A) = 2$.

We now state the main result of the paper.

Theorem 2.7. *Let Co_1 be the Conway's largest sporadic simple group. Then*

(a) rank$(Co_1 : nX) = 3$ *if* $nX \in \{2A, 2B, 2C, 3A\}$.

(b) rank$(Co_1 : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 2C, 3A\}$.

Proof. The proof follows from Lemmas 2.1, 2.2, …, and 2.5. □

References

[1] F. Ali and M. A. F. Ibrahim, On the ranks of *HS* and *McL*, Utilitas Mathematica, (2005). (To appear).

[2] F. Ali and M. A. F. Ibrahim, On the ranks of Co_2 and Co_3. (Submitted).

[3] M. Aschbacher, *Sporadic Groups*, Cambridge Univ. Press, London-New York, 1994.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Oxford Univ. Press, Eynsham, 1985.

[5] M. D. E. Conder, R. A. Wilson and A. J. Woldar, The symmetric genus of sporadic groups, Proc. Amer. Math. Soc. **116** (1992), no. 3, 653–663.

[6] M. R. Darafsheh and A. R. Ashrafi, $(2, p, q)$-generations of the Conway group Co_1, Kumamoto J. Math. **13** (2000), 1–20.

[7] M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, (p, q, r)-generations of the Conway group Co_1 for odd p, Kumamoto J. Math. **14** (2001), 1–20.

[8] M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, nX-complementary generations of the sporadic group Co_1, Acta Math. Vietnam. **29** (2004), no. 1, 57–75.

[9] S. Ganief and J. Moori, Generating pairs for the Conway groups Co_2 and Co_3, J. Group Theory **1** (1998), no. 3, 237–256.

[10] J. I. Hall and L. H. Soicher, Presentations of some 3-transposition groups, Comm. Algebra **23** (1995), no. 7, 2517–2559.

[11] I. M. Isaacs, *Character theory of finite groups*, Corrected reprint of the 1976 original [Academic Press, New York], Dover, New York, 1994.

[12] J. Moori, Generating sets for F_{22} and its automorphism group, J. Algebra **159** (1993), no. 2, 488–499.

[13] J. Moori, Subgroups of 3-transposition groups generated by four 3-transpositions, Quaestiones Math. **17** (1994), no. 1, 83–94.

[14] J. Moori, On the ranks of the Fischer group F_{22}, Math. Japon. **43** (1996), no. 2, 365–367.

[15] J. Moori, On the ranks of Janko groups J_1, J_2 and J_3, in *41st annual congress of South African Mathematical Society*, RAU, Auckland Park, (1998). (Private Communication).

[16] The GAP Group, **GAP** - Groups, Algorithms and programming, version 4.3, Aachen, St Andrews, 2003. (http://www-gap.dcs.st-and.ac.uk/~gap).

[17] L. L. Scott, Matrices and cohomology, Ann. of Math. (2) **105** (1977), no. 3, 473–492.

[18] R. A. Wilson, The maximal subgroups of Conway's group Co_1, J. Algebra **85** (1983), no. 1, 144–165.

[19] A. J. Woldar, Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus, Comm. Algebra **18** (1990), no. 1, 15–86.