Weighted L^p Sobolev-Lieb-Thirring inequalities

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Abstract: We give a weighted L^p version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions.

Key words: Sobolev-Lieb-Thirring inequalities; A_p-weights.

1. Introduction. In 1976 Lieb and Thirring proved the following inequality.

Theorem 1.1 ([4]). *Let $n \in \mathbf{N}$. Then there exists a positive constant c_n such that for every family $\{\phi_i\}_{i=1}^N$ in $H^1(\mathbf{R}^n)$ which is orthonormal in $L^2(\mathbf{R}^n)$, we have*

$$(1) \quad \int_{\mathbf{R}^n} \left(\sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} dx \le c_n \sum_{i=1}^N \|\nabla \phi_i\|^2.$$

In this theorem $H^1(\mathbf{R}^n)$ denotes the Sobolev space and $\|\cdot\|$ is the norm of $L^2(\mathbf{R}^n)$. In [4] Lieb and Thirring applied this inequality to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved a generalization of (1) under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbf{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^N \xi_i \overline{\xi_j} (\phi_i, \phi_j) \le \sum_{i=1}^N |\xi_i|^2$$

holds for all $\xi_i \in \mathbf{C}$, $i = 1, \dots, N$, where $(\cdot, \cdot)$ means the L^2 inner product ([2]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations. In this paper we shall give a weighted L^p version of (1) under the suborthonormal condition on $\{\phi_i\}$.

For the statement of our result we need to recall the definition of A_p-weights (c.f. [3, 5]). By a cube in $\mathbf{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let w be a non-negative, locally integrable function on $\mathbf{R}^n$. We say that w is an A_p-weight for $1 < p < \infty$ if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{p-1} \le C$$

for all cubes $Q \subset \mathbf{R}^n$. For example, $w(x) = |x|^\alpha$ is an A_p-weight when $-n < \alpha < n(p-1)$.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(y)\, dy \le C w(x) \quad \text{a.e.} \quad x \in Q$$

for all cubes $Q \subset \mathbf{R}^n$. If $-n < \alpha \le 0$, then $w(x) = |x|^\alpha$ is an A_1-weight. Let A_p be the class of A_p-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

A nonnegative, locally integrable function w on $\mathbf{R}^n$ is called a weight function. For a weight function w we define

$$L^p(w) = \left\{ f\colon \text{measurable on } \mathbf{R}^n, \int_{\mathbf{R}^n} |f(x)|^p w(x)\, dx < \infty \right\}.$$

The following is a conclusion of [7, Theorem 1.2] and [6, Lemma 3.2].

Theorem 1.2. *Let $n \in \mathbf{N}$, $3 \le n$, $w \in A_2$, and $w^{-n/2} \in A_{n/2}$. Then there exists a positive constant c such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbf{R}^n)$ which is suborthonormal in $L^2(\mathbf{R}^n)$ and $|\nabla \phi_i| \in L^2(w)$, $i = 1, \dots, N$, we have*

$$\int_{\mathbf{R}^n} \left(\sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} w(x)\, dx \le c \sum_{i=1}^N \int_{\mathbf{R}^n} |\nabla \phi_i(x)|^2 w(x)\, dx,$$

where c depends only on n and w.

By using this theorem we can prove the following weighted L^p version of the Sobolev-Lieb-Thirring inequality.

Theorem 1.3. *Let $n \in \mathbf{N}$ and $3 \le n$. Let $2n/(n+2) < p < n$, $p \ne 2$, and w be a weight function. When $p > 2$, we assume that $w^{n/(n-p)} \in$*

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$A_{p(n-2)/(2(n-p))}$. *When* $p < 2$, *we assume that* $w^{n/(n-2)} \in A_1$.

Then there exists a positive constant c such that for every family $\{\phi_i\}_{i=1}^N$ *in* $L^2(\mathbf{R}^n)$ *which is sub-orthonormal in* $L^2(\mathbf{R}^n)$ *and* $|\nabla\phi_i| \in L^p(w)$, $i = 1, \ldots, N$, *we have*

$$\int_{\mathbf{R}^n} \left(\sum_{i=1}^N |\phi_i(x)|^2\right)^{(1+2/n)p/2} w(x)\,dx \leq c\int_{\mathbf{R}^n} \left(\sum_{i=1}^N |\nabla\phi_i(x)|^2\right)^{p/2} w(x)\,dx,$$

where c depends only on n, p *and* w.

This is a new result even in the case $w \equiv 1$. When $2 < p < n$, an example of w is given by $w(x) = |x|^\alpha$, $-n + p < \alpha < n(p-2)/2$. When $2n/(n+2) < p < 2$, an example of w is given by $w(x) = |x|^\alpha$, $-n + 2 < \alpha \leq 0$.

2. Proof of Theorem 1.3. Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where f is a locally integrable function on $\mathbf{R}^n$ and the supremum is taken over all cubes Q which contain x. The following proposition is proved in [3, Chapter IV] or [5, Chapter V].

Proposition 2.1. (i) *Let* $1 < p < \infty$ *and* w *be a weight function on* $\mathbf{R}^n$. *Then there exists a positive constant c such that*

$$\int_{\mathbf{R}^n} M(f)^p w\,dx \leq c\int_{\mathbf{R}^n} |f|^p w\,dx$$

for all $f \in L^p(w)$ *if and only if* $w \in A_p$.

(ii) *Let* $1 < p < \infty$ *and* $w \in A_p$. *Then there exists a* $q \in (1, p)$ *such that* $w \in A_q$.

(iii) *Let* $0 < \tau < 1$ *and f be a locally integrable function on* $\mathbf{R}^n$ *such that* $M(f)(x) < \infty$ *a.e. Then* $M(f)^\tau \in A_1$.

(iv) *Let* $1 < p < \infty$. *Then* $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$, *where* $p^{-1} + p'^{-1} = 1$.

(v) *Let* $1 < p < \infty$ *and* $w_1, w_2 \in A_1$. *Then* $w_1 w_2^{1-p} \in A_p$.

Proof of Theorem 1.3. Our proof is very similar to that of the extrapolation theorem by Rubio de Francia (c.f. [1, Theorem 7.8]). In our proof the integral means that over $\mathbf{R}^n$.

Let $2 < p < n$ and $2/p + 1/q = 1$. We remark that the assumption $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$ leads to $w \in A_p$ by an easy calculation. Let $u \in L^q(w)$, $u \geq 0$, and $\|u\|_{L^q(w)} = 1$.

Since $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$, we have $w^{-2/(p-2)} \in A_{p(n-2)/(n(p-2))}$ by (iv) of Proposition 2.1. Hence there exists a γ such that $n/(n-2) < \gamma < q$ and $w^{-2/(p-2)} \in A_{p/(\gamma(p-2))}$ by (ii) of Proposition 2.1. Then we have $uw \leq M((uw)^\gamma)^{1/\gamma}$ a.e. Because

$$w^{-2q/p} = w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} = A_{q/\gamma}$$

and

$$\int M((uw)^\gamma)^{q/\gamma} w^{-2q/p}\,dx \leq c\int (uw)^q w^{-2q/p}\,dx = c\int u^q w\,dx = c \tag{2}$$

by (i) of Proposition 2.1, we get $M((uw)^\gamma)(x) < \infty$ a.e. Hence $M((uw)^\gamma)^{1/\gamma} \in A_1$ by (iii) of Proposition 2.1. Let $\alpha = n/((n-2)\gamma)$. Then $0 < \alpha < 1$ and

$$M((uw)^\gamma)^{-n/(2\gamma)} = \{M((uw)^\gamma)^\alpha\}^{1-n/2} \in A_{n/2},$$

where we used $M((uw)^\gamma)^\alpha \in A_1$ and (v) of Proposition 2.1. Let

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

Then we have

$$\begin{aligned}
&\int \rho^{1+2/n} uw\,dx \\
&\leq \int \rho^{1+2/n} M((uw)^\gamma)^{1/\gamma}\,dx \\
&\leq c\int \left(\sum_{i=1}^N |\nabla\phi_i|^2\right) M((uw)^\gamma)^{1/\gamma}\,dx \\
&\leq c\left(\int \left(\sum_{i=1}^N |\nabla\phi_i|^2\right)^{p/2} w\,dx\right)^{2/p} \\
&\qquad \times \left(\int M((uw)^\gamma)^{q/\gamma} w^{-2q/p}\,dx\right)^{1/q} \\
&\leq c\left(\int \left(\sum_{i=1}^N |\nabla\phi_i|^2\right)^{p/2} w\,dx\right)^{2/p}
\end{aligned}$$

where we used Theorem 1.2 and (2). If we take the supremum for all $u \in L^q(w)$, $u \geq 0$, and $\|u\|_{L^q(w)} = 1$, then we get

$$\left(\int \rho^{(1+2/n)p/2} w\,dx\right)^{2/p}$$

$$\leq c \left(\int \left(\sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{p/2} w\,dx \right)^{2/p}.$$

Next we consider the case $2n/(n+2) < p < 2$. We remark that $w \in A_1$ by the assumption $w^{n/(n-2)} \in A_1$. Let

$$f = \left(\sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{1/2}.$$

We can take γ such that $(2-p)n/2 < \gamma < p$. Then

$$\int M(f^\gamma)^{p/\gamma} w\,dx \leq c \int f^p w\,dx < \infty,$$

where we used $w \in A_1 \subset A_{p/\gamma}$ and (i) of Proposition 2.1. Hence we have $M(f^\gamma)(x) < \infty$ a.e. and

$$M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1$$

by (iii) of Proposition 2.1. Furthermore we have

$$M(f^\gamma)^{-(2-p)/\gamma} w \in A_2,$$

where we used

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1, \quad w \in A_1,$$

and (v) of Proposition 2.1. Moreover

$$\begin{aligned} &\{M(f^\gamma)^{-(2-p)/\gamma} w\}^{-n/2} \\ &= M(f^\gamma)^{(2-p)n/(2\gamma)} (w^{n/(n-2)})^{(1-n/2)} \in A_{n/2} \end{aligned}$$

because $w^{n/(n-2)} \in A_1$. Therefore

$$\begin{aligned} &\int \rho^{(1+2/n)p/2} w\,dx \\ &= \int \rho^{(1+2/n)p/2} w M(f^\gamma)^{-(2-p)p/(2\gamma)} \\ &\quad \times M(f^\gamma)^{(2-p)p/(2\gamma)}\,dx \\ &\leq \left(\int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma} w\,dx \right)^{p/2} \\ &\quad \times \left(\int M(f^\gamma)^{p/\gamma} w\,dx \right)^{1-p/2} \\ &\leq c \left(\int f^2 M(f^\gamma)^{-(2-p)/\gamma} w\,dx \right)^{p/2} \\ &\quad \times \left(\int f^p w\,dx \right)^{1-p/2} \\ &\leq c \left(\int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma} w\,dx \right)^{p/2} \\ &\quad \times \left(\int f^p w\,dx \right)^{1-p/2} \\ &\leq c \left(\int M(f^\gamma)^{p/\gamma} w\,dx \right)^{p/2} \left(\int f^p w\,dx \right)^{1-p/2} \\ &\leq c \int f^p w\,dx, \end{aligned}$$

where we used Theorem 1.2 in the second inequality.

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