Equisingularity in $\boldsymbol{R}^2$ as Morse stability in infinitesimal calculus

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Abstract: Two seemingly unrelated problems are intimately connected. The first is the equsingularity problem in $\mathbf{R}^2$: For an analytic family $f_t : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$, when should it be called an "equisingular deformation"? This amounts to finding a suitable trivialization condition (as strong as possible) and, of course, a criterion. The second is on the Morse stability. We define $\mathbf{R}_*$, which is $\mathbf{R}$ "enriched" with a class of infinitesimals. How to generalize the Morse Stability Theorem to polynomials over $\mathbf{R}_*$? The space $\mathbf{R}_*$ is much smaller than the space used in Non-standard Analysis. Our infinitesimals are analytic arcs, represented by fractional power series. In our Theorem II, (B) is a trivialization condition which can serve as a definition for equisingular deformation; (A), and (A′) in Addendum 1, are criteria, using the stability of "critical points" and the "complete initial form"; (C) is the Morse stability (Remark 1.6).

Key words: Morse equisingularity; infinitesimals; Newton Polygon.

1. Results. As in the Curve Selection Lemma, by a *parameterized arc* at 0 in $\mathbf{R}^2$ (resp. $\mathbf{C}^2$) we mean a *real* analytic map germ $\vec{\lambda} : [0, \epsilon) \to \mathbf{R}^2$ (resp. $\mathbf{C}^2$), $\vec{\lambda}(0) = 0$, $\vec{\lambda}(s) \not\equiv 0$. We call the image set, $\boldsymbol{\lambda} := \mathrm{Im}(\vec{\lambda})$, a (geometric) *arc* at 0, or the *locus* of $\vec{\lambda}$; call $\vec{\lambda}$ *a parametrization* of $\boldsymbol{\lambda}$.

Take $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. The distance from $P \in \boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ is a fractional power series in $s := \overline{OP}$, $dist(P, \boldsymbol{\mu}) = as^h + \cdots$, where $a > 0$, $h \in \mathbf{Q}^+$.

We call $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\mu}) := h$ the ***contact order*** of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Define $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}) := \infty$.

Let $\mathbf{S}^1_*$, or simply $\mathbf{S}_*$, denote the set of arcs at 0 in $\mathbf{R}^2$. This is called the *enriched unit circle* for the following reason. The tangent half line at 0, $\boldsymbol{l}$, of a given $\boldsymbol{\lambda}$ can be identified with a point of the unit circle $\mathbf{S}^1$. If $\boldsymbol{\lambda} \neq \boldsymbol{l}$, then $1 < \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{l}) < \infty$. Hence we can regard $\boldsymbol{\lambda}$ as an "*infinitesimal*" *at* $\boldsymbol{l}$, and $\mathbf{S}_*$ as $\mathbf{S}^1$ *"enriched"* with infinitesimals.

Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ be analytic. Write $V_*^{\mathbf{C}}(f) := \{\boldsymbol{\zeta} \in \mathbf{S}^3_* \mid f(z, w) \equiv 0 \text{ on } \boldsymbol{\zeta}\}$, where $\mathbf{S}^3_*$ denotes the set of arcs at 0 in $\mathbf{C}^2 (= \mathbf{R}^4)$, and $f(z, w)$ is the complexification of f.

For $\boldsymbol{\lambda} \in \mathbf{S}_*$, write $\mathcal{O}(\boldsymbol{\lambda}, V_*^{\mathbf{C}}(f)) := \max\{\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\zeta}) \mid \boldsymbol{\zeta} \in V_*^{\mathbf{C}}(f)\}$. Define the ***f-height*** of $\boldsymbol{\lambda}$ by $h_f(\boldsymbol{\lambda}) := \mathcal{O}(\boldsymbol{\lambda}, V_*^{\mathbf{C}}(f))$. Hence $h_f(\boldsymbol{\lambda}) = \infty$ if $f(x, y) \equiv 0$ along $\boldsymbol{\lambda}$.

For $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2 \in \mathbf{S}_*$, define $\boldsymbol{\lambda}_1 \sim_f \boldsymbol{\lambda}_2$ *if and only if* $h_f(\boldsymbol{\lambda}_1) = h_f(\boldsymbol{\lambda}_2) < \mathcal{O}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$. (In fact, $h_f(\boldsymbol{\lambda}_1) < \mathcal{O}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ implies $h_f(\boldsymbol{\lambda}_1) = h_f(\boldsymbol{\lambda}_2)$.) The equivalence class of $\boldsymbol{\lambda}$ is denoted by $\boldsymbol{\lambda}_f$.

We call $\boldsymbol{\lambda}_f$ an ***f-truncated arc***, or simply an ***f-arc***. Write $\mathbf{S}_{*/f} := \mathbf{S}_*/\sim_f$, $h(\boldsymbol{\lambda}_f) := h_f(\boldsymbol{\lambda})$.

Define the ***contact order*** of $\boldsymbol{\lambda}_f$ and $\boldsymbol{\mu}_f$ by: if $\boldsymbol{\lambda}_f \neq \boldsymbol{\mu}_f$, $\mathcal{O}(\boldsymbol{\lambda}_f, \boldsymbol{\mu}_f) := \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $\boldsymbol{\lambda} \in \boldsymbol{\lambda}_f$, $\boldsymbol{\mu} \in \boldsymbol{\mu}_f$; and $\mathcal{O}(\boldsymbol{\lambda}_f, \boldsymbol{\lambda}_f) := \infty$. This is well-defined. Write $\mathcal{O}(\boldsymbol{\lambda}_f, V_*^{\mathbf{C}}(f)) := \mathcal{O}(\boldsymbol{\lambda}, V_*^{\mathbf{C}}(f))$.

From now on we assume $f(x, y)$ is ***mini-regular*** in x, that is, regular in x of order $m(f)$, the multiplicity of f.

Let $\mathbf{R}^+_*$ (resp. $\mathbf{R}^+_{*/f}$) denote those arcs of $\mathbf{S}_*$ (resp. $\mathbf{S}_{*/f}$) in $y > 0$, not tangent to the x-axis, and $\mathbf{R}^-_*$ (resp. $\mathbf{R}^-_{*/f}$) denote those in $y < 0$. Write $\mathbf{R}_* := \mathbf{R}^+_* \cup \mathbf{R}^-_*$, $\mathbf{R}_{*/f} := \mathbf{R}^+_{*/f} \cup \mathbf{R}^-_{*/f}$.

Take $\boldsymbol{\lambda}_f$, $\boldsymbol{\mu}_f \in \mathbf{R}^+_{*/f}$, or $\in \mathbf{R}^-_{*/f}$. Define $\boldsymbol{\lambda}_f \simeq \boldsymbol{\mu}_f$ (read: "bar equivalent") *if and only if* either $\boldsymbol{\lambda}_f = \boldsymbol{\mu}_f$, or else $h(\boldsymbol{\lambda}_f) = h(\boldsymbol{\mu}_f) = \mathcal{O}(\boldsymbol{\lambda}_f, \boldsymbol{\mu}_f)$. Call an equivalence class an ***f-bar***. The one containing $\boldsymbol{\lambda}_f$ is denoted by $B(\boldsymbol{\lambda}_f)$, having ***height*** $h(B(\boldsymbol{\lambda}_f)) := h(\boldsymbol{\lambda}_f)$. (See [3–5].)

If $h(\boldsymbol{\lambda}_f) = \infty$ then $B(\boldsymbol{\lambda}_f) = \{\boldsymbol{\lambda}_f\}$, a singleton, and conversely.

The given coordinates (x, y) yield a coordinate on each bar of finite height, as follows:

Take B, say in $\mathbf{R}^+_{*/f}$, $h(B) < \infty$. Take $\boldsymbol{\lambda} \in \boldsymbol{\lambda}_f \in$

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B with parametrization $\vec{\lambda}(s)$. Eliminating s ($s \geq 0$) yields a *unique* fractional power series (as in [7])

$$(1) \qquad x = \lambda(y) = a_1 y^{\frac{n_1}{d}} + a_2 y^{\frac{n_2}{d}} + \cdots, \quad d \leq n_1 < n_2 < \cdots, \quad (y \geq 0).$$

Here all $a_i \in \mathbf{R}$. Let $\lambda_B(y)$ denote $\lambda(y)$ with all terms y^e, $e \geq h(B)$, deleted. Observe that for any $\boldsymbol{\mu} \in \boldsymbol{\lambda}_f \in B$, $\mu(y)$ has the form $\mu(y) = \lambda_B(y) + uy^{h(B)} + \cdots$, where $u \in \mathbf{R}$ is *uniquely* determined by $\boldsymbol{\lambda}_f$. We say $\boldsymbol{\lambda}_f \in B$ has ***canonical coordinate*** u, writing $\boldsymbol{\lambda}_f := u$. We call $x = \lambda_B(y)$, which depends only on B, the ***canonical representation*** of B.

Take B, $h(B) < \infty$, and $u = \boldsymbol{\lambda}_f \in B$. Let us write

$$f(\lambda_B(y) + uy^{h(B)} + \cdots, y) := I_f^B(u) y^e + \cdots, \quad I_f^B(\boldsymbol{\lambda}_f) := I_f^B(u) \neq 0.$$

An important observation is that e depends only on B, not on $\boldsymbol{\lambda}_f$; $I_f^B(u)$ depends only on $\boldsymbol{\lambda}_f$, not on $\boldsymbol{\lambda} \in \boldsymbol{\lambda}_f$, and is a polynomial (Lemma 1.2 below). We call $L_f(B) := L_f(\boldsymbol{\lambda}_f) := e$ the ***Lojasiewicz exponent*** of f on B.

Attention/Convention. *Not every $u \in \mathbf{R}$ is a canonical coordinate. For example, $f(x,y) = x^2 - y^3$ has a bar B of height $3/2$, and ± 1 are not canonical coordinates; $I_f^B(u)$ is not a priori defined at ± 1. Since I_f^B is a polynomial, we shall regard it as defined for all $u \in \mathbf{R}$.*

In general, the canonical coordinate identifies B with a copy of $\mathbf{R}$ minus the real roots of I_f^B. Hence $\bar{B}$, the metric space completion, is a copy of $\mathbf{R}$.

If $B = \{\boldsymbol{\lambda}_f\}$, a singleton, we define $I_f^B(\boldsymbol{\lambda}_f) := 0$, $L_f(\boldsymbol{\lambda}_f) := \infty$.

Now, take $l(x,y) := x$, and consider $\mathbf{S}_{*/l}$. If $\nu(y) = ay^e + \cdots$, $a \neq 0$, $e \geq 1$, then the l-arc $\boldsymbol{\nu}_l$ can be identified with $(a, e) \in (\mathbf{R} - \{0\}) \times \mathbf{Q}^{+1}$, $\mathbf{Q}^{+1} := \{r \in \mathbf{Q}^+ \mid r \geq 1\}$. If $\nu(y) \equiv 0$ then $h(\boldsymbol{\nu}_l) = \infty$; we write $\boldsymbol{\nu}_l := (0, \infty)$. We call $\mathcal{V} := ((\mathbf{R} - \{0\}) \times \mathbf{Q}^{+1}) \cup \{(0,\infty)\} (= \mathbf{R}^{\pm}_{*/l})$ the ***infinitesimal value space***. The given f, mini-regular in x, induces a $\mathcal{V}$-valued function

$$f_* : \mathbf{R}_{*/f} \to \mathcal{V}, \quad f_*(\boldsymbol{\lambda}_f) := \left(I_f^B(\boldsymbol{\lambda}_f), L_f(\boldsymbol{\lambda}_f)\right) \in \mathcal{V}, \quad (\boldsymbol{\lambda}_f \in B).$$

Take $z \in \mathbf{C}$. We say z is a B-***root*** of f if f has a Newton-Puiseux root of the form $\alpha(y) = \lambda_B(y) + zy^{h(B)} + \cdots$. The number of such roots is the *multiplicity* of z.

Definition 1.1. Take $c := \boldsymbol{\gamma}_f \in B$. If $h(B) < \infty$ and $c (\in \mathbf{R})$ is a B-root of f_x, say of multiplicity k, we say $\boldsymbol{\gamma}_f$ is a (*real*) ***critical point*** of f_* of multiplicity $m(\boldsymbol{\gamma}_f) := k$.

If $B = \{\boldsymbol{\gamma}_f\}$, and $m(B) \geq 2$, we also call $\boldsymbol{\gamma}_f$ a critical point of multiplicity $m(B) - 1$.

Call $f_*(c) := f_*(\boldsymbol{\gamma}_f) \in \mathcal{V}$ the ***critical value*** at $\boldsymbol{\gamma}_f$.

If f_x has complex B-root(s), but no real B-root, then we take a *generic* real number r, put $\gamma(y) := \lambda_B(y) + ry^{h(B)}$, and call $\boldsymbol{\gamma}_f$ *the* real critical point in B with multiplicity $m(\boldsymbol{\gamma}_f) := 1$. (Convention: For different such B, we take *different* generic r.)

The above is the list of all (real) critical points. (If f_x has no B-root, B yields no critical point.) The number of critical points is finite (Lemma 1.2).

Now, let $\mathbf{M}$ be the maximal ideal of $\mathbf{R}\{s\}$, furnished with the point-wise convergence topology, that is, the smallest topology so that the projection maps

$$\pi_N : \mathbf{M} \longrightarrow \mathbf{R}^N, \quad a_1 s + \cdots + a_N s^N + \cdots \mapsto (a_1, \ldots, a_N), \quad N \in \mathbf{Z}^+,$$

are continuous. Furnish $\mathbf{S}_*$, $\mathbf{S}_{*/f}$ with the quotient topologies by the quotient maps

$$p_* : \mathbf{M}^2 - \{0\} \to \mathbf{S}_*, \quad p_{*/f} : \mathbf{M}^2 - \{0\} \to \mathbf{S}_{*/f}.$$

Take $\vec{\lambda} \in \mathbf{M}^2$, and a real-valued function, α, defined near $\vec{\lambda}$. We say α is *analytic* at $\vec{\lambda}$ if $\alpha = \varphi \circ \pi_N$, π_N a projection, φ an analytic function at $\pi_N(\vec{\lambda})$ in $\mathbf{R}^N$. This defines an analytic structure on $\mathbf{M}^2$. We furnish $\mathbf{S}_*$ and $\mathbf{S}_{*/f}$ with the quotient analytic structure.

In the following, let I be a sufficiently small neighborhood of 0 in $\mathbf{R}$. We write "c-" for "continuous", "a-" for "analytic", "c/a-" for "continuous (resp. analytic)".

Let $F(x,y;t)$ be a given t-parameterized a-deformation of $f(x,y)$. That is to say, $F(x,y;t)$ is real analytic in (x,y,t), defined for (x,y) near $0 \in \mathbf{R}^2$, $t \in I$, with $F(x,y;0) = f(x,y)$, $F(0,0;t) \equiv 0$. When t is fixed, we also write $F(x,y;t)$ as $f_t(x,y)$.

In $\mathbf{S}_* \times I$ define $(\boldsymbol{\lambda}, t) \sim_F (\boldsymbol{\lambda}', t')$ *if and only if* $t = t'$ and $\boldsymbol{\lambda} \sim_{f_t} \boldsymbol{\lambda}'$. Denote the quotient space by $\mathbf{S}_* \times_F I$. Similarly, $\mathbf{R}^{\pm}_* \times_F I := \mathbf{R}^{\pm}_* \times I / \sim_F$.

By a t-parameterized ***c/a-deformation*** of $\boldsymbol{\lambda}_f$ we mean a family of f_t-arcs, $\boldsymbol{\lambda}_{f_t}$, obtained as follows. Take a parametrization $\vec{\lambda}(s)$ of $\boldsymbol{\lambda}_f$, and a c/a-map: $I \to \mathbf{M}^2$, $t \mapsto \vec{\lambda}_t$, $\vec{\lambda}_0 = \vec{\lambda}$. Then $\boldsymbol{\lambda}_{f_t} := p_{*/f_t}(\vec{\lambda}_t)$.

This is equivalent to taking a c/a-map: $I \to \mathbf{S}_* \times_F I$, $t \mapsto (\boldsymbol{\lambda}_{f_t}, t)$. A ***c/a-deformation*** of a given B is, *by definition*, a family $\{B_t\}$ obtained by taking any $\boldsymbol{\lambda}_f \in B$, a c/a-deformation $\boldsymbol{\lambda}_{f_t}$, and then $B_t := B(\boldsymbol{\lambda}_{f_t})$.

Theorem I. *The following three conditions are equivalent.*

(**a**) *Each (real) critical point, $\boldsymbol{\gamma}_f$, of f_* is **stable along** $\{f_t\}$ in the sense that $\boldsymbol{\gamma}_f$ admits a c-deformation $\boldsymbol{\gamma}_{f_t}$, a critical point of $(f_t)_*$, such that $m(\boldsymbol{\gamma}_{f_t})$, $h(\boldsymbol{\gamma}_{f_t})$, $L_{f_t}(\boldsymbol{\gamma}_{f_t})$ are constants. (If $\boldsymbol{\gamma}_f$ arises from the generic number r, we use the same r for $\boldsymbol{\gamma}_{f_t}$.)*

(**b**) *There exists a (t-level preserving) homeomorphism*

$$H : (\mathbf{R}^2 \times I, 0 \times I) \to (\mathbf{R}^2 \times I, 0 \times I),$$
$$((x, y), t) \mapsto (\eta_t(x, y), t),$$

which is bi-analytic off the t-axis $\{0\} \times I$, with the following five properties:

(b.1) *$f_t(\eta_t(x, y)) = f(x, y)$, $t \in I$, (trivialization of $F(x, y; t)$);*

(b.2) *Given any bar B, $\eta_t(\vec{\alpha}(s))$ is analytic in $(\vec{\alpha}, s, t)$, $\vec{\alpha} \in p^{-1}_{*/f}(B)$ (analyticity on each bar); in particular, η_t is arc-analytic, for any fixed t;*

(b.3) *$\mathcal{O}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{O}(\eta_t(\boldsymbol{\alpha}), \eta_t(\boldsymbol{\beta}))$ (contact order preserving); moreover, $\eta_t(\boldsymbol{\alpha}_f) \in \mathbf{S}_{*/f_t}$ is well-defined (invariance of truncated arcs).*

(b.4) *The induced mapping $\eta_t : B \to B_t$ extends to an analytic isomorphism: $\bar{B} \to \bar{B}_t$.*

(b.5) *If c is a critical point of f_*, then $c_t = \eta_t(c)$ is one of $(f_t)_*$, $m(c) = m(c_t)$.*

(**c**) *There exists an **isomorphism** $H_* : \mathbf{R}_{*/f} \times I \to \mathbf{R}_* \times_F I$, $(\boldsymbol{\alpha}_f, t) \mapsto (\eta_t(\boldsymbol{\alpha}_f), t)$, preserving critical points and multiplicities. That is to say, H_* is a homeomorphism,*

(c.1) *Given B, $B_t := \eta_t(B)$ is a bar, $h(B_t) = h(B)$, $m(B_t) = m(B)$;*

(c.2) *The restriction of η_t to B extends to an analytic isomorphism $\bar{\eta}_t : \bar{B} \to \bar{B}_t$;*

(c.3) *If c is a critical point of f_*, then $c_t := \eta_t(c)$ is one of $(f_t)_*$, $m(c) = m(c_t)$.*

Theorem II. *The following three conditions are equivalent.*

(**A**) *The function f_* is **Morse stable along** $\{f_t\}$. That is, every critical point is stable along $\{f_t\}$, and for critical points $c \in B$, $c' \in B'$, $f_*(c) = f_*(c')$ implies $(f_t)_*(c_t) = (f_t)_*(c'_t)$.*

(**B**) *There exists H, as in (**b**), with an additional property*:

(b.6) *If c, c' are critical points, $f_*(c) = f_*(c')$, then $(f_t)_*(c_t) = (f_t)_*(c'_t)$.*

(**C**) *There exist an isomorphism H_* as in (**c**), and an isomorphism $K_* : \mathcal{V} \times I \to \mathcal{V} \times I$, such that $K_* \circ (f_* \times id) = \Phi \circ H_*$, where $\Phi(\boldsymbol{\alpha}_{f_t}, t) := ((f_t)_*(\boldsymbol{\alpha}_{f_t}), t)$.*

Lemma 1.2. *Let $\{z_1, \dots, z_q\}$ be the set of B-roots of f ($z_i \in \mathbf{C}$), $h(B) < \infty$. Then*

$$I^B_f(u) = a \prod_{i=1}^{q} (u - z_i)^{m_i},$$

$0 \neq a \in \mathbf{R}$, *a constant, m_i the multiplicity of z_i.*

In particular, $I^B_f(u)$ is a polynomial with real coefficients.

If $c := \boldsymbol{\gamma}_f \in B$ is a critical point of f_*, then $\frac{d}{du} I^B_f(c) = 0 \neq I^B_f(c)$, and conversely. The multiplicity of c (as a critical point of the polynomial $I^B_f(u)$) equals $m(\boldsymbol{\gamma}_f)$.

The number of critical points of f_* in $\mathbf{R}^+_{*/f}$ (resp. $\mathbf{R}^-_{*/f}$) is bounded by $m(f) - 1$.

Definition 1.3. The degree of $I^B_f(u)$ is called the ***multiplicity*** of B, denoted by $m(B)$.

We say B is a ***polar bar*** if $I^B_f(u)$ has at least two distinct roots (in $\mathbf{C}$), or B is a singleton with $m(B) \geq 2$. Call $\mathcal{I}(f) := \{(B, I^B_f) \mid B \text{ polar}\}$ the ***complete initial form*** of f.

Corollary 1.4. *Each critical point belongs to a polar bar; each polar bar contains at least one critical point.*

We recall Morse Theory. Take an a-family of real polynomials $p_t(x) = a_0(t)x^d + \cdots + a_d(t)$, $a_0(0) \neq 0$, $t \in I$, as an a-deformation of $p(x) := p_0(x)$. Let $c_0 \in \mathbf{R}$ be a critical point of $p(x)$, of multiplicity $m(c_0)$. We say c_0 is *stable along* $\{p_t\}$, if it admits a c-deformation c_t, $\frac{d}{dx} p_t(c_t) = 0$, $m(c_t) = m(c_0)$. (A c-deformation c_t, if exists, is necessarily an a-deformation.)

Definition 1.5. We say $p(x)$ is ***Morse and zero stable*** along $\{p_t\}$ if:

(i) Every (real) critical point of $p_0(x)$ is stable along $\{p_t\}$;

(ii) For critical points c_0, c'_0, $p_0(c_0) = p_0(c'_0)$ implies $p_t(c_t) = p_t(c'_t)$.

(iii) If $p_0(c_0) = \frac{d}{dx} p_0(c_0) = 0$, then $p_t(c_t) = \frac{d}{dx} p_t(c_t) = 0$.

Remark 1.6. Theorem II generalizes in spirit a version of the Morse Stability Theorem : If $p(x)$ is Morse and zero stable along $\{p_t\}$ then there exist analytic isomorphisms $H, K : \mathbf{R}\times I \to \mathbf{R}\times I$, such that $K\circ(p\times id) = \Phi\circ H$, $K(0,t) \equiv 0$, where $\Phi(x,t) := (p_t(x), t)$.

That (a) $\Rightarrow$ (c) reduces to the following. Given $x = f_i(t)$, $1 \le i \le N$, analytic, $f_i(t) \ne f_j(t)$, for $i \ne j$, $t \in I$. There exists an analytic isomorphism $H : \mathbf{R}\times I \to \mathbf{R}\times I$, $(x,t) \mapsto (\eta_t(x), t)$, $\eta_t(f_i(t)) = const$, $1 \le i \le N$. (Proved by Cartan's Theorem A, or Interpolation.)

We say $\mathcal{I}(f)$ is ***Morse and zero stable*** along $\{f_t\}$ if each polar B admits a c-deformation B_t, a polar bar of f_t, such that two of $h(B_t)$, $m(B_t)$, $L_{f_t}(B_t)$ are constants (we can then show all three are), and $\{I_f^B\}$ is Morse and zero stable along $\{I_{f_t}^{B_t}\}$, for each B.

Addendum 1. (**B**) *is also equivalent to* (**A′**): $\mathcal{I}(f)$ *is Morse and zero stable along* $\{f_t\}$.

2. Relative Newton polygons. Take $\boldsymbol{\lambda}$, say in $\mathbf{R}_*^+$, with $\lambda(y)$. Let us change variables: $X := x - \lambda(y)$, $Y := y$,

$$\mathcal{F}(X,Y) := f(X+\lambda(Y),Y) := \sum a_{ij}X^iY^{j/d},\quad i,j\ge 0,\ i+j>0.$$

In the first quadrant of a coordinate plane we plot a dot at $(i, j/d)$ for each $a_{ij} \ne 0$, called a (Newton) dot. The Newton polygon of $\mathcal{F}$ in the usual sense is called the *Newton Polygon of f relative to* $\boldsymbol{\lambda}$, denoted by $\mathbf{P}(f,\boldsymbol{\lambda})$. (See [4].) Write $m_0 := m(f)$. Let the vertices be

$$V_0 = (m_0, 0), \dots, V_k = (m_k, q_k),\quad q_i \in \mathbf{Q}^+,\ m_i > m_{i+1},\ q_i < q_{i+1}.$$

The (Newton) *edges* are: $E_i = \overline{V_{i-1}V_i}$, with *angle* θ_i, $\tan\theta_i := \frac{q_i - q_{i-1}}{m_{i-1}-m_i}$, $\pi/4 \le \theta_i < \pi/2$; a vertical one, E_{k+1}, sitting at V_k, $\theta_{k+1} = \pi/2$; a horizontal one, E_0, which is unimportant.

If $m_k \ge 1$ then $f \equiv 0$ on $\boldsymbol{\lambda}$. If $m_k \ge 2$, f is singular on $\boldsymbol{\lambda}$. If $\boldsymbol{\lambda} \sim_f \boldsymbol{\lambda}'$ then $\mathbf{P}(f,\boldsymbol{\lambda}) = \mathbf{P}(f,\boldsymbol{\lambda}')$, hence $\mathbf{P}(f,\boldsymbol{\lambda}_f)$ is well-defined.

Notation: $L(E_i) := \overline{V_{i-1}V_i'}$, $V_i' := (0, q_{i-1} + m_{i-1}\tan\theta_i)$, *i.e.* E_i extended to the y-axis.

Fundamental Lemma. *Suppose each polar bar B admits a c-deformation B_t such that $h(B_t)$ and $m(B_t)$ are independent of t. Then each $\boldsymbol{\lambda}_f \in \mathbf{R}_{*/f}$ admits an a-deformation $\boldsymbol{\lambda}_{f_t} \in \mathbf{R}_{*/f_t}$ such that $\mathbf{P}(f_t,\boldsymbol{\lambda}_{f_t})$ is independent of t. The induced deformation $B_t := B(\boldsymbol{\lambda}_{f_t})$ of $B_0 := B(\boldsymbol{\lambda}_f)$, and hence the a-deformation $x = \lambda_{B_t}(y)$ of the canonical representation $x = \lambda_{B_0}(y)$, are uniquely defined; that is, if we take any $\boldsymbol{\eta}_f \in B(\boldsymbol{\lambda}_f)$, and a c-deformation $\boldsymbol{\eta}_{f_t}$ with $\mathbf{P}(f_t,\boldsymbol{\eta}_{f_t}) = \mathbf{P}(f,\boldsymbol{\lambda}_f)$, then $B(\boldsymbol{\eta}_{f_t}) = B(\boldsymbol{\lambda}_{f_t})$.*

Given B, B'. The contact order $\mathcal{O}(B_t, B_t')$, defined below, is independent of t.

For $B \ne B'$, define $\mathcal{O}(B,B') := \mathcal{O}(\boldsymbol{\lambda}_f, \boldsymbol{\lambda}'_f)$, $\boldsymbol{\lambda}_f \in B$, $\boldsymbol{\lambda}'_f \in B'$; and $\mathcal{O}(B,B) := \infty$.

The Lemma is proved by a succession of Tschirnhausen transforms at the vertices, beginning at V_0, which represents $a_{m0}X^m$ in $\mathcal{F}(X,Y)$, $m := m(f)$. Let us define $\mathcal{P}$ by

$$(2)\quad F(X+\lambda(Y),Y;t) := \mathcal{F}(X,Y) + \mathcal{P}(X,Y;t),\quad \mathcal{P}(X,Y;t) := \sum p_{ij}(t)X^iY^{j/d},$$

where $p_{ij}(t)$ are analytic, $p_{ij}(0) = 0$. Take a root of $\frac{\partial^{m-1}}{\partial X^{m-1}}[a_{m0}X^m + \mathcal{P}(X,Y;t)] = 0$,

$$X = \rho_t(Y) := \sum b_j(t)Y^{j/d},\quad b_j(0) = 0,$$

$b_j(t)$ analytic. (Implicit Function Theorem.)

Thus, $\lambda(y) + \rho_t(y)$ is an a-deformation of $\lambda(y)$. Let $X_1 := X - \rho_t(Y)$, $Y_1 := Y$. Then

$$F(X_1 + \lambda(Y_1) + \rho_t(Y_1), Y_1; t) := \mathcal{F}(X_1,Y_1) + \mathcal{P}^{(1)}(X_1,Y_1;t),$$

where $\mathcal{P}^{(1)} := \sum p_{ij}^{(1)}(t)X_1^iY_1^{j/d}$, $p_{ij}^{(1)}(0) = 0$, and $p_{m-1,j}^{(1)}(t) \equiv 0$ (Tschirnhausen).

For brevity, we shall write the coordinates (X_1, Y_1, t) simply as (X, Y, t), abusing notations. That is, we now have $p_{m-1,j}(t) \equiv 0$ in (2).

We claim that $\mathcal{P}$ in fact has no dot below $L(E_1)$. This is proved by contradiction.

Suppose it has. Take a generic number $s \in \mathbf{R}$. Let $\zeta(y) := \lambda(y) + sy^e$, $e := \tan\theta_1$, and

$$F(\widetilde{X} + \zeta(\widetilde{Y}), \widetilde{Y}; t) := \mathcal{F}(\widetilde{X},\widetilde{Y}) + \widetilde{\mathcal{P}},\quad \widetilde{\mathcal{P}}(\widetilde{X},\widetilde{Y};0) \equiv 0.$$

Since s is generic, $\mathbf{P}(f,\boldsymbol{\zeta}_f)$ has only one edge, which is $L(E_1)$, and $B(\boldsymbol{\zeta}_f)$ is polar. Below $L(E_1)$, $\widetilde{\mathcal{P}}$ has at least one dot (when $t \ne 0$), but still no dot of the form $(m-1, q)$.

A c-deformation B_t of $B(\boldsymbol{\zeta}_f)$ would either create new dot(s) of the form $(m-1,q)$ below $L(E_1)$, or else not change the existing dot(s) of $\widetilde{\mathcal{P}}$ below $L(E_1)$. (This is the spirit of the Tschirnhausen transformation.) Thus, as $t \ne 0$, $h(B_t)$ or $m(B_t)$, or both, will drop. This contradicts to the hypothesis of the Fundamental Lemma.

This argument can be repeated recursively at V_1, V_2, etc., to clear all dots under $\mathbf{P}(f,\boldsymbol{\lambda}_f)$. More precisely, suppose in (2), $\mathcal{P}$ has no dots below $L(E_i)$,

$0 \le i \le r$. By the Newton-Puiseux Theorem, there exists a root ρ_t of $\frac{\partial^{m_r-1}}{\partial X^{m_r-1}}[aX^{m_r}Y^{q_r} + \mathcal{P}] = 0$ with $\mathcal{O}_y(\rho_t) \ge \tan\theta_{r+1}$, where $aX^{m_r}Y^{q_r}$ is the term for V_r. A Tschirnhausen transform will then eliminate all dots of $\mathcal{P}$ of the form $(m_r - 1, q)$. As before, all dots below $L(E_{r+1})$ also disappear.

We have seen the *only* way to clear dots below $\mathbf{P}(f, \boldsymbol{\lambda}_f)$ is by the Tschirnhausen transforms. If $\mathbf{P}(f, \boldsymbol{\eta}_{f_t}) = \mathbf{P}(f, \boldsymbol{\lambda}_f)$, we must have $\mathcal{O}(\boldsymbol{\lambda}_{f_t}, \boldsymbol{\eta}_{f_t}) \ge h(B_0)$. The uniqueness follows.

Define a partial ordering "$>$" by: $B > \hat{B}$ if and only if $h(B) > h(\hat{B}) = \mathcal{O}(\boldsymbol{\lambda}_f, \boldsymbol{\mu}_f)$, $\boldsymbol{\lambda}_f \in B$, $\boldsymbol{\mu}_f \in \hat{B}$. Let $\hat{B}$ be the largest bar so that $B \ge \hat{B}$, $B' \ge \hat{B}$. We write $\lambda_B(y) = \lambda_{\hat{B}}(y) + ay^e + \cdots$, $\lambda_{B'}(y) = \lambda_{\hat{B}}(y) + by^e + \cdots$, $e := h(\hat{B})$. The uniqueness of $\hat{B}_t$ completes the proof.

3. Vector fields. Assume (**a**). We use a vector field $\vec{v}$ to prove (**b**). The other implications are not hard.

Take a critical point $\boldsymbol{\gamma}_f$, say in B, $\gamma(y) = \lambda_B(y) + cy^{h(B)}$. Let B_t be the deformation of B. Let c_t be the a-deformation of c, $\frac{d}{du}I^{B_t}_{f_t}(c_t) = 0$, $m(c_t) = m(c)$. (If c is generic, take $c_t = c$.)

Let $\gamma_t(y) := \lambda_{B_t}(y) + c_t y^{h(B_t)}$. Then $\boldsymbol{\gamma}_t$ is a critical point of f_t in B_t.

Now, let $\boldsymbol{\gamma}^{(i)}_f$, $1 \le i \le N$, denote all the critical points of f, for *all* (polar) B. For brevity, write $\boldsymbol{\gamma}^{(i)} := \boldsymbol{\gamma}^{(i)}_f$, with deformations $\boldsymbol{\gamma}^{(i)}_t$, just defined.

We can assume $F(x, 0; t) = \pm x^m$, and hence $\frac{\partial F}{\partial t}(x, 0; t) \equiv 0$. As $F(x, 0; t) = a(t)x^m + \cdots$, $a(0) \ne 0$, a substitution $u = \sqrt[m]{|a(t)|} \cdot x + \cdots$ will bring $F(x, 0, t)$ to this form.

We can also assume $\boldsymbol{\gamma}^{(i)} \in \mathbf{R}^+_{*/f}$ for $1 \le i \le r$, and $\boldsymbol{\gamma}^{(i)} \in \mathbf{R}^-_{*/f}$ for $r + 1 \le i \le N$.

For each $\boldsymbol{\gamma}^{(i)} \in \mathbf{R}^+_{*/f}$, we now construct a vector field $\vec{v}^+_i(x, y, t)$, defined for $y \ge 0$.

Write $\boldsymbol{\gamma}_t := \boldsymbol{\gamma}^{(i)}_t$. Let $X := x - \gamma_t(y), Y := y$. Then $\mathcal{F}(X, Y; T) := F(X + \gamma_t(Y), Y; T)$ is analytic in $(X, Y^{1/d}, T)$. As in [1, 6], define $\vec{v}^+_i(x, y, t) := \vec{V}(x - \gamma_t(y), y, t)$, $y \ge 0$, where

$$(3) \quad \vec{V}(X, Y, t) := \frac{X\mathcal{F}_X\mathcal{F}_t}{(X\mathcal{F}_X)^2 + (Y\mathcal{F}_Y)^2} \cdot X\frac{\partial}{\partial X} + \frac{Y\mathcal{F}_Y\,\mathcal{F}_t}{(X\mathcal{F}_X)^2 + (Y\mathcal{F}_Y)^2} \cdot Y\frac{\partial}{\partial Y} - \frac{\partial}{\partial t}.$$

In general, given $\boldsymbol{\alpha}_i$, $x = \alpha_i(y)$, say in $\mathbf{R}^+_*$, $1 \le i \le r$. Let $q(x, y) := \prod_{k=1}^r (x - \alpha_k(y))^2$,

$$q_i(x, y) := q(x, y)/(x - \alpha_i(y))^2,$$

$$p_i(x, y) := q_i(x, y)/[q_1(x, y) + \cdots + q_r(x, y)].$$

We call $\{p_1, \dots, p_r\}$ a ***partition of unity*** for $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_r\}$.

Now, take $\{p_i\}$ for $\{\boldsymbol{\gamma}^{(1)}_t, \cdots \boldsymbol{\gamma}^{(r)}_t\}$. Define $\vec{v}^+(x, y, t) := \sum_{i=1}^r p_i(x, y, t)\,\vec{v}^+_i(x, y, t)$.

Similarly, $\boldsymbol{\gamma}^{(i)}_f$, $r + 1 \le i \le N$, yield $\vec{v}^-(x, y, t)$, $y \le 0$. We can then glue $\vec{v}^\pm(x, y, t)$ together along the x-axis, since $\vec{v}^\pm(x, 0, t) \equiv -\frac{\partial}{\partial t}$. This is our vector field $\vec{v}(x, y, t)$, which, by (3), is clearly tangent to the level surfaces of $F(x, y; t)$, proving (b.1).

4. Sketch of Proof.

Lemma 4.1. *Let $W(X, Y)$ be a weighted form of degree d, $w(X) = h$, $w(Y) = 1$. Take u_0, not a multiple root of $W(X, 1)$. If $W(u_0, 1) \ne 0$ or $u_0 \ne 0$ then, with $X = uv^h$, $Y = v$,*

$$|XW_X| + |YW_Y| = \mathit{unit} \cdot |\,v\,|^d, \text{ for } u \text{ near } u_0.$$

For, by Euler's Theorem, if $X - u_0Y^h$ divides W_X and W_Y, then u_0 is a multiple root.

To show (b.2), etc., take $\boldsymbol{\alpha}$, say in $\mathbf{R}^+_*$. Take k, $\mathcal{O}(\boldsymbol{\gamma}^{(k)}, \boldsymbol{\alpha}) = \max\{\mathcal{O}(\boldsymbol{\gamma}^{(j)}, \boldsymbol{\alpha}) \mid 1 \le j \le r\}$.

We can assume $\boldsymbol{\alpha}$ is not a multiple root of f, $e := \mathcal{O}(\boldsymbol{\gamma}^{(k)}, \boldsymbol{\alpha}_f) < \infty$. (If $\boldsymbol{\alpha}$ is, then $\boldsymbol{\gamma}^{(k)} = \boldsymbol{\alpha}_f$, $h(B) = \infty$. This case is easy.)

Write $B := B(\boldsymbol{\alpha}_f)$ if $B(\boldsymbol{\alpha}_f) \le B(\boldsymbol{\gamma}^{(k)})$, and $B := B(\boldsymbol{\gamma}^{(k)})$ if $B(\boldsymbol{\alpha}_f) > B(\boldsymbol{\gamma}^{(k)})$.

Thus $\alpha(y) = \lambda_B(y) + ay^e + \cdots$, $\frac{d}{du}I^B_f(a) \ne 0$. Let us consider the mapping

$$\tau : (u, v, t) \mapsto (x, y, t) := (\lambda_{B_t}(v) + uv^e, v, t),$$
$$u \in \mathbf{R},\ 0 \le v < \varepsilon,\ t \in I,$$

B_t the deformation of B, and the liftings $\vec{\nu}^+_j := (d\tau)^{-1}(p_j\vec{v}^+_j)$, $\vec{\nu}^+ := \sum_{j=1}^r \vec{\nu}^+_j$.

Key Lemma. *The lifted vector fields $\vec{\nu}^+_j$, and hence $\vec{\nu}^+$, are analytic at (u, v, t), if u is not a multiple root of $I^{B_t}_{f_t}$. Moreover, $\vec{\nu}^+(u, 0, t)$ is analytic for all $u \in \mathbf{R}$; that is, $\lim_{v\to 0^+} \vec{\nu}^+(u, v, t)$ has only removable singularities on the u-axis.*

We analyze each $\vec{\nu}^+_i$, using (3). For brevity, write $\mathbf{B} := B(\boldsymbol{\gamma}^{(i)})$, $\mathbf{B}_t := B(\boldsymbol{\gamma}^{(i)}_t)$.

First, consider the case $B = \mathbf{B}$. This case exposes the main ideas.

Now I^B_f and $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$ are related as follows. Let $W(X, Y) = \sum_{i,j} a_{ij}X^iY^{j/d}$ be the (unique) weighted form such that $W(u, 1) = I^B_f(u + c)$, $w(X) = h(B)$, $w(Y) = 1$, where c is the canonical coordinate of $\boldsymbol{\gamma}^{(i)}$. The Newton dots on the highest compact edge of $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$ represent the non-zero terms of $W(X, Y)$; the highest vertex is $(0, L_f(B))$.

Thus $\frac{d}{du}W(0,1) = \frac{d}{du}I_f^B(c) = 0$, $W(0,1) \neq 0$. The weighted degree of $W(X,Y)$ is $L_f(B)$.

Hence, by Lemma 4.1, the substitution $X = x - \lambda_B(y) - cy^{h(B)} = (u-c)v^{h(B)}$, $Y = v$, yields $\mathcal{O}_v(|X\mathcal{F}_X| + |Y\mathcal{F}_Y|) = L_f(\mathbf{B})$, if $u - c$ is not a multiple root of $W(u,1)$.

The Newton Polygon is independent of t: $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)}) = \mathbf{P}(f_t, \boldsymbol{\gamma}_t^{(i)})$. All Newton dots of $\mathcal{F}$, and hence those of $\mathcal{F}_T$, are contained in $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$. Hence $\mathcal{O}_v(\mathcal{F}_T((u-c)v^{h(B)}, v; T)) \geq L_f(B)$.

By the Chain Rule, we have $X\frac{\partial}{\partial X} = (u-c)\frac{\partial}{\partial u}$, $Y\frac{\partial}{\partial Y} = v\frac{\partial}{\partial v} - h(B)(u-c)\frac{\partial}{\partial u}$.

It follows that $(d\tau)^{-1}(\vec{v}_i^+)$ and $\vec{\nu}_i$ are analytic at (u,v,t), if u is not a multiple root of $I_{f_t}^{B_t}$.

Next, suppose $B < \mathbf{B}$. Again we show $(d\tau)^{-1}(\vec{v}_i^+)$ has the required property.

Write $\gamma^{(i)}(y) := \lambda_B(y) + c'y^{h(B)} + \cdots$. Let $W(X,Y)$ denote the weighted form such that $W(u,1) = I_f^B(u + c')$, $w(X) = h(B)$, $w(Y) = 1$.

If $W(X,Y)$ has more than one terms, they are dots on a compact edge of $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$, not the highest one. If $W(X,Y)$ has only one term, it is a vertex, say $(\bar{m}, \bar{q})$, $\bar{m} \geq 2$.

In either case, $u = 0$ is a multiple root of $W(u,1)$. All Newton dots of $\mathcal{F}_T$ are contained in $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$. The rest of the argument is the same as above.

Finally, suppose $B \not\leq \mathbf{B}$. Here p_i plays a vital role in analyzing $\vec{\nu}_i^+$.

Let $\bar{B}$ denote the largest bar such that $B > \bar{B} \leq \mathbf{B}$.

Let $U := x - \lambda_{B_t}(y)$, $V := y$. The identity $p_i = p_k q_i / q_k$, and the Chain Rule yield

$$p_i \cdot X\frac{\partial}{\partial X} = p_k \frac{(U+\varepsilon)^2}{(U+\delta)^2}(U+\delta)\frac{\partial}{\partial U},$$
$$p_i \cdot Y\frac{\partial}{\partial Y} = p_k \cdot \frac{(U+\varepsilon)^2}{(U+\delta)^2}\left[V\frac{\partial}{\partial V} - V\delta'(V)\frac{\partial}{\partial U}\right],$$

where $\delta := \delta(y,t) := \lambda_{B_t}(y) - \gamma_t^{(i)}(y)$, $\varepsilon := \lambda_{B_t}(y) - \gamma_t^{(k)}(y)$, $\mathcal{O}_y(\delta) = h(\bar{B}) < h(B) \leq \mathcal{O}_y(\varepsilon)$.

The substitution $U = uv^{h(B)}$, $V = v$ lifts both to analytic vector fields in (u,v,t).

It remains to study $\Psi := \mathcal{F}_T / (|X\mathcal{F}_X| + |Y\mathcal{F}_Y|)$ when $X = \delta(v,t) + uv^{h(B)}$, $Y = v$.

Let $\mathcal{G}(U,V,T) := \mathcal{F}(U + \delta(V,T), V, T)$. The Chain Rule yields

(4) $X\mathcal{F}_X = (U+\delta)\mathcal{G}_U$, $Y\mathcal{F}_Y = V(\mathcal{G}_V - \delta_V\mathcal{G}_U)$,
$\mathcal{F}_T = \mathcal{G}_T - \delta_T\mathcal{G}_U$.

Let us compare $\mathbf{P}(f, \boldsymbol{\gamma}^{(i)})$ and $\mathbf{P}(\mathcal{G}, U = 0)$, the (usual) Newton Polygon of $\mathcal{G}$. Let E_i', θ_i' and $V_i{}'$ denote the edges, angles and vertices of the latter. Then $E_i = E_i'$, for $1 \leq i \leq l$, where l is the largest integer such that $\tan\theta_l < h(\bar{B})$. Moreover, $\theta'_{l+1} = \theta_{l+1}$ (although E_{l+1}, E'_{l+1} may be different).

Consider the vertex $V'_{l+1} := (m'_{l+1}, q'_{l+1})$, $m'_{l+1} \geq 2$. It yields a term $\mu := a(T)U^pV^q$ of $\delta\mathcal{G}_U$, $a(0) \neq 0$, $p := m'_{l+1} - 1$, $q := q'_{l+1} + \tan\theta_{l+1}$. With the substitution $U = uv^{h(B)}$, $(u \neq 0,)$ $V = v$, μ is the dominating term in (4). That is, $\mathcal{O}_v(\mu) < \mathcal{O}_v(\mu')$, for all terms μ' in $U\mathcal{G}_U$, $V\mathcal{G}_V$, etc., (and for all terms $\mu' \neq \mu$ in $\delta\mathcal{G}_U$), since $\mathcal{O}_Y(\delta) = \tan\theta_{l+1}$.

It follows that Ψ is analytic. That $\lim \vec{\nu}_i^+$ has only removable singularities also follows.

Conditions (b.2) etc. can be derived from the Key Lemma.

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