$\mathcal{D}$-modules associated to the determinantal singularities

By Philibert NANG

Institute of Mathematics, University of Tsukuba
1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571
(Communicated by Heisuke HIRONAKA, M. J. A., May 12, 2004)

Abstract: We describe a classification of regular holonomic $\mathcal{D}$-modules whose characteristic variety is the union of conormal bundles to the orbits of the linear group. We show that the category of such objects is equivalent to the one of graded modules of finite type over an algebra of invariant differential operators with polynomial coefficients.

Key words: $\mathcal{D}$-modules; regular holonomic $\mathcal{D}$-modules; homogeneous sections; invariant sections; determinantal singularities; finite diagrams.

1. Introduction. This paper generalizes the classification given in [11]. Let X be the complex vector space of $n \times n$ matrices, $\mathcal{D}_X$ the sheaf of differential operators on it. Denote by G the quotient group of $GL_n(\mathbf{C}) \times GL_n(\mathbf{C})$ by the kernel of its action on X. There are $n+1$ orbits $X_k \subset X$: the subset of matrices of rank $k = 0, \cdots, n$. We are interested in the description of regular holonomic $\mathcal{D}_X$-modules whose characteristic variety is the union Λ of conormal bundles to these orbits. Denote by $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_X)$ this category. The main result is the following: let $\widetilde{G} := SL_n(\mathbf{C}) \times SL_n(\mathbf{C}) \times \mathbf{C}$ be the universal covering of G and $\mathcal{A}$ the algebra of $\widetilde{G}$-invariant differential operators acting on $\widetilde{G}$-invariant functions. Denote by $\theta \in \mathcal{A}$ the Euler vector field on X and $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbf{C}} \mathbf{C}[\theta]\, u < \infty$ $\forall u \in T$.

Theorem. *The categories* $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_X)$ *and* $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ *are equivalent.*

Note that a similar result with a different point of view has been obtained by [2]. Also results along these lines have been obtained by several mathematicians (see [1], [3], [9–11] etc.).

2. Homogeneous sections and the Weyl algebra.

2.1. Homogeneous sections. We will use the theory of analytic $\mathcal{D}$-modules developped in [4–8].

Definition 1. A section s of a $\mathcal{D}_X$-module is said to be homogeneous if $\dim_{\mathbf{C}} \mathbf{C}[\theta]\, s < \infty$. We say that s is homogeneous of degree $\lambda \in \mathbf{C}$ if there exists $j \in \mathbf{N}$ such that $(\theta - \lambda)^j s = 0$.

Theorem 2 [10, Theorem 1.3]**.** *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with a good filtration $(F_k\mathcal{M})_{k \in \mathbf{Z}}$ stable by θ. Then*

i) *$\mathcal{M}$ is generated by a finite number of global sections $(s_i)_{i=1,\cdots,p}$ in $\mathcal{M}$ such that $\dim_{\mathbf{C}} \mathbf{C}[\theta]\, s_i < \infty$,*

ii) *the vector space*
$$\Gamma(X, F_k\mathcal{M}) \bigcap \left[\bigcup_{p \in \mathbf{N}} \ker(\theta - \lambda)^p \right]$$
of homogeneous global sections of $F_k\mathcal{M}$ of degree λ is finite dimensional $\forall k \in \mathbf{Z}$, $\forall \lambda \in \mathbf{C}$.

2.2. Algebra of invariant differential operators. As usual $\mathcal{W}$ indicates the Weyl algebra on X. Let us determine the algebra $\overline{\mathcal{A}} := \Gamma(X, \mathcal{D}_X)^{\widetilde{G}} \subset \mathcal{W}$ of $\widetilde{G}$-invariant differential operators with polynomial coefficients. Let $x_1 = (x_{ij})$, $d_1 = {}^t(\partial/\partial x_{ij})$ be matrices with entries in $\mathcal{D}_X$. We shall denote by $x_m := \wedge^m x_1$ (resp. $d_m := \wedge^m d_1$) the exterior mth power of x_1 (resp. d_1) that is the $\binom{n}{m} \times \binom{n}{m}$ matrice of all $m \times m$ minors δ_K^I (resp. Δ_K^I) of x_1 (resp. d_1) where
$$\delta_K^I := \sum_{\pi \in \mathcal{S}_m} \mathrm{sgn}(\pi)\, x_{\pi(k(1))}^{i(1)} \cdots x_{\pi(k(m))}^{i(m)}$$
with $\mathcal{S}_m$ the symmetric group, $I = (i(1), \cdots, i(m))$ (resp. $K = (k(1), \cdots, k(m))$), $1 \le i(1) \le \cdots \le i(m) \le n$ (resp. $1 \le k(1) \le \cdots \le k(m) \le n$). In particular $x_{n-1} = \wedge^{n-1} x_1 = x_1^{-1} \det(x_1)$ (resp. d_{n-1}) is the adjoint matrice of x_1 (resp. d_1). The action of the group $\widetilde{G}$ on these matrices is defined as follows: for any $g = (a, b) \in \widetilde{G}$, we have $g \cdot$

2000 Mathematics Subject Classification. Primary 32C38; Secondary 32S25, 32S60.

$(x_1, d_1) = \left(ax_1b^{-1}, bd_1a^{-1}\right)$. Let us denote by Tr the trace map. We set $\delta := (1/n)\,\mathrm{Tr}\,x_1x_{n-1} = \det(x_1)$, $\Delta := (1/n)\,\mathrm{Tr}\,d_1d_{n-1} = \det(d_1)$, $\theta := \mathrm{Tr}\,x_1d_1$ (the Euler vector field on X) and $q_m = \mathrm{Tr}\,x_md_m$ for $m = 2, \cdots, n-1$. These are $\widetilde{G}$-invariant differential operators with polynomial coefficents. We have the following proposition:

Proposition 3. *The algebra $\overline{\mathcal{A}}$ is generated over $\mathbf{C}$ by the $n+1$ operators δ, Δ, θ, $q_2, q_3, \cdots, q_{n-1}$ such that $^\forall m, i, j = 2, \cdots, n-1$*

$$(1)\qquad [\theta, \delta] = n\delta, \quad [\theta, \Delta] = -n\Delta,$$
$$[\theta, q_m] = 0, \quad [q_i, q_j] = 0,$$
$$[\Delta, \delta] = \frac{q_{n-1}}{n} + (n-1)! \prod_{k=1}^{n-2} \left(\frac{\theta}{n} + k\right)\left(\frac{\theta}{n} + n\right).$$

Denote by $\mathcal{I} \subset \mathcal{W}$ the left ideal generated by infinitesimal generators of G. If we put $\mathcal{J} := \overline{\mathcal{A}} \cap \mathcal{I}$, then $\mathcal{J}$ is the two sided ideal of $\widetilde{G}$-invariant differential operators annihilating $\widetilde{G}$-invariant functions. Therefore the quotient $\mathcal{A} := \overline{\mathcal{A}}/\mathcal{J}$ is the algebra of $\widetilde{G}$-invariant differential operators acting on $\widetilde{G}$-invariant functions.

Corollary 4. *The quotient algebra $\mathcal{A}$ is generated over $\mathbf{C}$ by δ, Δ, θ satisfying the above relations and such that*

$$(2)\qquad \delta\Delta = \prod_{l=0}^{n-1} \left(\frac{\theta}{n} + l\right), \quad \Delta\delta = \prod_{l=1}^{n} \left(\frac{\theta}{n} + l\right).$$

3. Modules supported by matrices of rank $\leq n-1$. We denote by $\overline{X}_k$ the set of matrices of rank k or less ($k = 0, 1, \cdots, n-1$). We still denote by δ, the determinant map $\delta : X \longrightarrow \mathbf{C}$, $x \longmapsto \det(x)$. Then we have $\overline{X}_{n-1} := \{x \in X \ / \ \delta(x) = 0\}$. In this section, we study the $\mathcal{D}_X$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$ supported by $\overline{X}_k$. Such objects will be useful in the next section to prove that any $\mathcal{D}_X$-module in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$ is generated by $\widetilde{G}$-invariant homogeneous global sections.

3.1. Inverse image. Now, we denote by $\delta^+(\mathcal{N})$ the inverse image by δ of a $\mathcal{D}_C$-module $\mathcal{N}$. Let t be a coordinate of $\mathbf{C}$ and $\partial_t := (\partial/\partial t)$, I_n the identity matrice on X. The transfer module $\mathcal{D}_{X\overset{\delta}{\longrightarrow}\mathbf{C}}$ is generated by an element K subject to the relations

$$(3)\qquad \delta K = Kt, \qquad d_1K = x_{n-1}K\partial_t.$$

This last implies the following one:

$$(4)\qquad x_1d_1K = I_nKt\partial_t, \quad \theta K = nKt\partial_t.$$

$$(5)\qquad d_{n-1}K = x_1K\partial_t \prod_{l=1}^{n-2} (t\partial_t + l),$$
$$x_{n-1}d_{n-1}K = I_nK \prod_{l=0}^{n-2} (t\partial_t + l).$$

$$(6)\qquad q_{n-1}K = nK \prod_{l=0}^{n-2} (t\partial_t + l) \quad \text{and}$$
$$\Delta K = K\partial_t \prod_{l=1}^{n-1} (t\partial_t + l).$$

The tranfer module $\mathcal{D}_{X\overset{\delta}{\longrightarrow}\mathbf{C}}$ is flat over $\delta^{-1}(\mathcal{D}_\mathbf{C})$ so the inverse image functor δ^+ is exact. If $\mathcal{N}$ is a regular holonomic $\mathcal{D}_\mathbf{C}$-module with singularity at $t = 0$, then the inverse image $\delta^+\mathcal{N}$ decomposes as $\mathcal{N}$. If the operator of multiplication by t is invertible on the $\mathcal{D}_\mathbf{C}$-module $\mathcal{N}$, then δ is invertible on $\delta^+\mathcal{N}$. In particular in this case any meromorphic section defined on $X\backslash\overline{X}_{n-1}$ extends to the whole X.

3.2. Study of $\delta^+(\mathcal{O}_\mathbf{C}(1/t))$. Now, we are interested in the sub $\mathcal{D}_X$-modules of $P := \delta^+(\mathcal{O}_\mathbf{C}(1/t)) = \mathcal{O}_X(1/\delta)$. The $\mathcal{D}_X$-module P is generated by its $\widetilde{G}$-invariant homogeneous global sections $e_k = Kt^k = \delta^kK$, $^\forall k \in \mathbf{Z}_{\leq 0}$. We have,

$$(7)\qquad \delta e_k = e_{k+1}, \quad d_1e_k = x_{n-1}e_{k-1},$$
$$d_{n-1}e_k = \prod_{l=0}^{n-2} (k+l)\, x_1e_{k-1}.$$

$$(8)\qquad q_{n-1}e_k = n \prod_{l=0}^{n-2} (k+l)\, e_k,$$
$$\Delta e_k = \prod_{l=0}^{n-1} (k+l)\, e_{k-1}.$$

The $\mathcal{D}_X$-module P has $(n+1)$ sub $\mathcal{D}_X$-modules P_{-j}, generated respectively by e_{-j} ($j = 0, 1, \cdots, n$). Denote by P^{-j} the quotient module associated to P_{-j} : $P^0 = P_0$; $P^{-j} := P_{-j}/P_{-j+1}$ if $j = 1, \cdots, n$. As in [11] (Section 3.2), the quotient P^{-j} is an irreducible holonomic $\mathcal{D}_X$-module of multiplicity 1 whose microsupport is $\Lambda_{n-j} := \overline{T^*_{X_{n-j}}X}$.

Therefore with the help of the relations (7), (8) and the fact that the P^{-j} are irreducible modules, we see that if $\mathcal{N}$ is a sub $\mathcal{D}_X$-module of P which is not contained in P_{-j} then $\mathcal{N}$ contains P_{-j+1}. Thus we get the following result.

Lemma 5. *$P_0, P_{-1}, \cdots, P_{-n}$ are the only sub $\mathcal{D}_X$-modules of P.*

Next we obtain the following

Proposition 6. *Any section* $s \in \Gamma(X \setminus \overline{X}_{j-1}, P_{-j})$ *of the* $\mathcal{D}_X$*-module* P_{-j} *in the complementary of* $\overline{X}_{j-1}$ *extends to the whole* X $(j = 1, \cdots, n-1)$.

Proof. The $\mathcal{D}_X$-module P_{-j} is the union of modules $\mathcal{O}_X e_{-k}$ $(0 \le k \le j)$ such that the associated graded modules $\operatorname{gr}(P_{-j})$ is the sum of modules $\mathcal{O}_{T^*_{\overline{X}_{n-j}}X} \widetilde{e}_{-k}$ $(0 \le k \le j)$. In this case the property of extension here is true for functions because $\overline{X}_j$ is normal along $\overline{X}_{j-1}$ $(j = 1, \cdots, n-1)$. □

4. Invariant sections. In this section, we see that any object $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$ is generated by its $\widetilde{G}$-invariant homogeneous global sections. This result will play a crucial role in the proof of our main theorem. To this end, first we show that there exists a $\mathcal{D}_C$-module $\mathcal{N}$ such that $\mathcal{M}$ is isomorphic to $\delta^+\mathcal{N}$ in $X \setminus \overline{X}_{n-2}$. Let

$$i : \mathbf{C} \longrightarrow X, \ t \longmapsto \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

be a section of the determinant map $\delta : X \longrightarrow \mathbf{C}$, $x \longmapsto \det(x)$ i.e. $\delta \circ i(\mathbf{C}) = \mathbf{C}$. Denote by $D := i(\mathbf{C})$ its image. The line D is non characteristic for $\mathcal{M}$ i.e. $\overline{T^*_D X} \bigcap \operatorname{char}(\mathcal{M}) \subset T^*_X X$. Then $\mathcal{M}$ is canonically isomorphic to $\delta^+ i^+(\mathcal{M})$ in the neighborhood of D. We know from [4] that the sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta^+ i^+(\mathcal{M}))$ is constructible. Also $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta^+ i^+(\mathcal{M}))$ is locally constant on the fibers $\delta^{-1}(t)$, $t \in \mathbf{C}$. Denote by u its canonical section defined in the neighborhood of D (corresponding with the isomorphism $\mathcal{M} \xrightarrow{\sim} \delta^+ i^+(\mathcal{M})$ which induces the identity on D). As in [11] (Proposition 4.1), by virtue of the simple connexity of the fibers $\delta^{-1}(t)$, $t \neq 0 \in \mathbf{C}$, we get the following result.

Proposition 7. *The isomorphism* $u : \mathcal{M}_{|D} \xrightarrow{\sim} \delta^+ i^+(\mathcal{M})_{|D}$ *extends to* $X \setminus \overline{X}_{n-2}$.

Now we should note that the infinitesimal action of G on X lifts to an action of the universal covering $\widetilde{G}$ on $\mathcal{M}$ (see [10, Proposition 1.6]). Let us prove the following theorem.

Theorem 8. $\mathcal{M}$ *is generated by its* $\widetilde{G}$*-invariant homogenoeus global sections.*

Proof. Denote by $\mathcal{M}^G \subset \mathcal{M}$ the submodule generated over $\mathcal{D}_X$ by $\widetilde{G}$-invariant homogeneous global sections. Let us show successively that the quotient module $\mathcal{M}/\mathcal{M}^G$ is supported by $\overline{X}_j$ $(j = 0, 1, \cdots, n-1)$. To begin with, $\mathcal{M}/\mathcal{M}^G$ is supported by $\overline{X}_{n-1}$: indeed $\mathcal{M}$ is isomorphic in $X \setminus \overline{X}_{n-1}$ to a $\mathcal{D}_X$-module $\delta^+\mathcal{N}$ (see Proposition 7). We may assume that the operator of multiplication by t is invertible on $\mathcal{N}$ such that there exists a morphism $v : \mathcal{M} \longrightarrow \delta^+\mathcal{N}$ which is an isomorphism out of $\overline{X}_{n-1}$ (see Section 3.1). The image $v(\mathcal{M})$ is a submodule of $\delta^+\mathcal{N}$ so it is generated by its $\widetilde{G}$-invariant global sections, and any invariant global sections s of a quotient of $\mathcal{M}$ lifts to an invariant section $\widetilde{s}$ of $\mathcal{M}$ $(\widetilde{s} \in \Gamma(X, \mathcal{M})^G)$. Therefore $\mathcal{M}/\mathcal{M}^G$ is supported by $\overline{X}_{n-1}$. Recall that $P_0, P_{-1}, \cdots, P_{-n}$ are the sub $\mathcal{D}_X$-modules of $P := \mathcal{O}_X(1/\delta)$ generated respectively by the invariant homogeneous sections e_{-j} $(j = 0, 1, \cdots, n)$ (see Section 3.2). If $\mathcal{M}$ is supported by $\overline{X}_{n-1}$, it is isomorphic out of $\overline{X}_{n-2}$ to a direct sum of copies of (P_{-n}/P_0), then there is a morphism $\mathcal{M} \longrightarrow (P_{-n}/P_0)^N$ whose sections extend such that $\mathcal{M}/\mathcal{M}^G$ is supported by $\overline{X}_{n-2}$ because the submodules of (P_{-n}/P_0) are also generated by their invariant sections. In the same way, if $\mathcal{M}$ is with support on $\overline{X}_{n-j}$ $(j = 2, \cdots, n-1)$ then there is a morphism $\mathcal{M} \longrightarrow (P_{-n}/P_{1-j})^N$ which is an isomorphism out of $\overline{X}_{n-j-1}$, such that $\mathcal{M}/\mathcal{M}^G$ is with support on $\overline{X}_{n-j-1}$ because the submodules of (P_{-n}/P_{1-j}) are also generated by their invariant sections. Finally, if $\mathcal{M}$ is supported by X_0 the result is obvious. □

5. Main theorem. As in Section 2.2, $\mathcal{W}$ is the Weyl algebra on X and $\overline{\mathcal{A}} \subset \mathcal{W}$ is the algebra of $\widetilde{G}$-invariant differential operators generated over $\mathbf{C}$ by δ, Δ, θ, q_j with $j = 2, \cdots, n-1$ satisfying the relations of Proposition 3. Also $\mathcal{A}$ is the quotient of $\overline{\mathcal{A}}$ by the two sided ideal of $\widetilde{G}$-invariant differential operators annihilating $\widetilde{G}$-invariant functions. The quotient algebra $\mathcal{A}$ is generated by δ, Δ, θ satisfying the relations (1) and such that

$$\delta\Delta = \prod_{l=0}^{n-1} \left(\frac{\theta}{n} + l\right) \quad \text{and} \quad \Delta\delta = \prod_{l=1}^{n} \left(\frac{\theta}{n} + l\right)$$

(see Corollary 4). Recall that $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ stands for the category of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbf{C}} \mathbf{C}[\theta] u < \infty$ $\forall u \in T$. If $\mathcal{M}$ is an object in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$, we denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma(X, \mathcal{M})$ consisting of $\widetilde{G}$-invariant global sections s of $\mathcal{M}$ such that $\dim_{\mathbf{C}} \mathbf{C}[\theta] s < \infty$. By virtue of Theorem 2 we see that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. Conversely if T is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, we set $\Phi(T) := \mathcal{M}_0 \bigotimes_{\mathcal{A}} T$ where $\mathcal{M}_0 := \mathcal{W}/\mathcal{I}$ with $\mathcal{I}$ the left ideal generated by infinitesimal generators of G. Then $\Phi(T)$ is an object in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$. Thus we have defined two functors

$$\Psi : \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \quad \text{and}$$
$$\Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X).$$

Theorem 9. *The functors Ψ and Φ induce an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}).$$

Proof. Let us denote by ε the canonical generator of $\mathcal{M}_0 := \mathcal{W}/\mathcal{I}$. If $h \in \mathcal{W}$, denote by $\widetilde{h} \in \overline{\mathcal{A}}$ its average over $SU_n(\mathbf{C}) \times SU_n(\mathbf{C})$. The average operator $\mathcal{W} \longrightarrow \overline{\mathcal{A}}$, $h \longmapsto \widetilde{h}$ induces a surjective morphism of $\mathcal{A}$-modules $v : \mathcal{M}_0 \longrightarrow \mathcal{A}$. More generally, for any T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v \otimes 1_T$ is a surjective map $v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A} \bigotimes_{\mathcal{A}} T = T$ which is the left inverse of $u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, $t \longmapsto \varepsilon \otimes t$. Thus u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T := \Phi(T)$ that is $\mathrm{Im}\, u_T = \Psi(\Phi(T))$. Therefore u_T is bijective from T to $\Psi(\Phi(T))$. Now consider the canonical morphism $w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}$. From Theorem 8 $\mathcal{M}$ is generated by its invariant sections $\Psi(\mathcal{M})$ thus w is surjective. The kernel $Q := \ker w$ is also generated by $\Psi(\mathcal{Q})$. Then $\Psi(\mathcal{Q}) \subset \Psi[\Phi(\Psi(\mathcal{M}))] = \Psi(\mathcal{M})$. Since $\Psi(\mathcal{M}) \longrightarrow \mathcal{M}$ is injective we obtain $\Psi(\mathcal{Q}) = 0$. Thus $Q = \mathcal{D}_X \Psi(\mathcal{Q}) = 0$ (see Theorem 8). Therefore w is bijective. □

5.1. Diagram associated to a graded module. In this subsection, we show that the objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ can be understood in terms of finite diagrams of linear maps. Indeed a graded $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ defines an infinite diagram consisting of finite dimensional vector spaces T_λ (with $(\theta - \lambda)$ being nilpotent on each T_λ, $\lambda \in \mathbf{C}$) and linear maps between them deduced from the multiplication by δ, Δ, θ homogeneous of degree n, $-n$, 0 respectively: $\cdots \rightleftarrows T_\lambda \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{\lambda+n} \rightleftarrows \cdots$ satisfying the relations of Corollary 4

$$\delta\Delta = \prod_{l=0}^{n-1} \left(\frac{\theta}{n} + l\right), \quad \Delta\delta = \prod_{l=1}^{n} \left(\frac{\theta}{n} + l\right).$$

Such a diagram is completely determined by a finite subset of objects and arrows. Indeed

a) For $\sigma \in \mathbf{C}/n\mathbf{Z}$, denote by $T^\sigma \subset T$ the submodule $T^\sigma = \bigoplus_{\lambda = \sigma \bmod n\mathbf{Z}} T_\lambda$. Then T is generated by the finite direct sum of T^σ's

$$T = \bigoplus_{\sigma \in \mathbf{C}/n\mathbf{Z}} T^\sigma = \bigoplus_{\sigma \in \mathbf{C}/n\mathbf{Z}} \left(\bigoplus_{\lambda = \sigma \bmod n\mathbf{Z}} T_\lambda \right).$$

b) If $\sigma \neq 0 \bmod n\mathbf{Z}$ ($\lambda = \sigma \bmod n\mathbf{Z}$), then the linear maps δ and Δ are bijective. Therefore T^σ is completely determined by one element T_λ.

c) If $\sigma = 0 \bmod n\mathbf{Z}$ ($\lambda = \sigma \bmod n\mathbf{Z}$), then T^σ is completely determined by a diagram of $n+1$ elements

$$T_{-n^2} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{-n(n-1)} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{-n(n-2)} \cdots \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{-n} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_0.$$

In the other degrees δ or Δ are bijective through the relations

$$\Delta\delta = \prod_{k=1}^{n} \left(\frac{\theta}{n} + k\right), \quad \delta\Delta = \prod_{k=0}^{n-1} \left(\frac{\theta}{n} + k\right).$$

Acknowledgements. We have greatly benefited from discussions with Prof. L. Boutet de Monvel. Special thanks are due to Prof. M. Kashiwara for bringing this problem to our attention and for his kind help.

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