A q-Mahler measure

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Abstract: We construct a q-analogue of the Mahler measure using a q-analogue of the logarithm. We present some basic examples, where we obtain q-analogues of special values of zeta functions. We calculate also the classical limit and the crystal limit.

Key words: Mahler measure; q-analogue; crystal; zeta function.

1. Introduction. The Mahler measure $m(f)$ of a polynomial $f(x_1,\dots,x_n)\in\mathbf{C}[x_1,\dots,x_n]$ is defined as

$$m(f) = \mathrm{Re}\int_0^1\cdots\int_0^1 \log\left(f(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})\right)d\theta_1\cdots d\theta_n.$$

It was originated in the study of transcendental numbers by Mahler [M] and thereafter it turned out to be related to various themes containing special values of zeta functions (Smyth [S], Boyd [B] and Deninger [D]) and to the entropy of the associated dynamical system (Lind-Schmidt-Ward [LSW]). We refer to Everest-Ward [EW] for a survey of the Mahler measure.

In this paper we present a q-analogue $m_q(f)$ of $m(f)$ for $q>1$. We recall that the q-analogue $l_q(x)$ of the logarithm function $\log(x)$ is defined by

$$l_q(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{[n]_q}$$

in $|x-1|<q$ originally, where $[n]_q=(q^n-1)/(q-1)$. Since $\lim_{q\downarrow 1}[n]_q=n$ we have

$$\lim_{q\downarrow 1} l_q(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}=\log(x).$$

(See [KC] for q-analogues.) Moreover, $l_q(x)$ has an analytic continuation to all $x\in\mathbf{C}$ as a meromorphic function via the expression

$$l_q(x)=(q-1)\sum_{m=1}^{\infty}\frac{x-1}{x-1+q^m}.$$

Actually, the definition implies

$$\begin{aligned} l_q(x) &= (q-1)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{q^n-1}\\ &= (q-1)\sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^n\sum_{m=1}^{\infty}q^{-nm}\\ &= (q-1)\sum_{m=1}^{\infty}\frac{(x-1)q^{-m}}{1+(x-1)q^{-m}}. \end{aligned}$$

Definition 1.1. Let $q>1$. Then the q-Mahler measure $m_q(f)$ of a polynomial $f(x_1,\dots,x_n)\in\mathbf{C}[x_1,\dots,x_n]$ is defined as

$$m_q(f) = \mathrm{Re}\int_0^1\cdots\int_0^1 l_q\left(f(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})\right)d\theta_1\cdots d\theta_n.$$

We report basic calculations.

Theorem 1.

$$m_q(x+a)=l_q(a)\quad \textit{for}\quad a>1.$$

Example 1.

$$m_3(x+2)=\sum_{m=1}^{\infty}\frac{1}{3^m+1}=l_3(2).$$

Theorem 2.

$$m_q((x+a)(x+b))=l_q(ab)\quad \textit{for}\quad a,b>1.$$

Example 2.

$$m_3((x+2)^2)=l_3(4)=2\sum_{m=0}^{\infty}\frac{1}{3^m+1}=l_3(2)+1.$$

Theorem 3.

$$m_q(x+y+1)=\frac{2(q-1)}{\pi}\sum_{m=1}^{\left[\frac{\log 2}{\log q}\right]}\cos^{-1}\left(\frac{q^m}{2}\right).$$

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Example 3.

$$m_{\sqrt{3}}(x+y+1)=\frac{\sqrt{3}-1}{3},\quad m_3(x+y+1)=0.$$

Taking the limit $q\downarrow 1$ in Theorem 1–3 we recover the classical basic results:

(1) $m(x+a)=\log(a)$ for $a>1$,

(2) $m((x+a)(x+b))=\log(ab)=\log(a)+\log(b)$ for $a,b>1$,

(3) $m(x+y+1)=\frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^2}=\frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3})$,

where χ_{-3} is the non-trivial Dirichlet character modulo 3 ((3) is the result due to Smyth [S]).

Cases (1) and (2) are easy to see the behavior as $q\downarrow 1$, but to obtain (3) from Theorem 3 is interesting in view of the quite different appearances.

A basic property of the usual Mahler measure is the additivity:

$$m(fg)=m(f)+m(g).$$

The following result shows counter examples for m_q; these are intimately related to the failure of $l_q(ab)=l_q(a)+l_q(b)$.

Theorem 4. (1) *Let* $|a|<q-1$. *Then*

$$m_q(ax)=-\sum_{n=1}^{\infty}\frac{1}{[n]_q}=l_q(0)<0.$$

(2) *When* $q>3$

$$m_q(2x)<m_q(2)+m_q(x).$$

(3) *When* $q>3$

$$m_q(2\cdot 2)>m_q(2)+m_q(2).$$

2. Proofs of theorem 1 and theorem 2. We start from

$$m_q(x+a)=(q-1)\,\mathrm{Re}\sum_{m=1}^{\infty}\int_0^1\frac{e^{2\pi i\theta}+a-1}{e^{2\pi i\theta}+a-1+q^m}d\theta.$$

Denote the integral by I_m. Then we have

$$I_m=\frac{1}{2\pi i}\oint\frac{z+a-1}{z+a-1+q^m}\cdot\frac{dz}{z}$$

with the integration on the unit circle $|z|=1$. Notice that $a>1$ implies $a-1+q^m>q^m>1$ for $m\geq 1$. Hence $z+a-1+q^m\neq 0$ in $|z|\leq 1$. Thus

$$I_m=\frac{a-1}{a-1+q^m}.$$

This gives Theorem 1.

In the case of Theorem 2 we look at

$$m_q((x+a)(x+b))=(q-1)\,\mathrm{Re}\sum_{m=1}^{\infty}\int_0^1\frac{(e^{2\pi i\theta}+a)(e^{2\pi i\theta}+b)-1}{(e^{2\pi i\theta}+a)(e^{2\pi i\theta}+b)-1+q^m}d\theta.$$

It is sufficient to show that the integral is equal to $(ab-1)/(ab-1+q^m)$. This follows from the fact that $(z+a)(z+b)-1+q^m\neq 0$ in $|z|\leq 1$, which is easily checked by using $a,b>1$.

3. Proof of theorem 3. We have

$$\begin{aligned}m_q(x+y+1)&=(q-1)\,\mathrm{Re}\sum_{m=1}^{\infty}\int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12}\frac{e^{2\pi i\theta_1}+e^{2\pi i\theta_2}}{e^{2\pi i\theta_1}+e^{2\pi i\theta_2}+q^m}d\theta_1 d\theta_2\\&=(q-1)\,\mathrm{Re}\sum_{m=1}^{\infty}\int_{-\frac12}^{\frac12}f_m(\theta)d\theta\end{aligned}$$

with

$$f_m(\theta)=\int_{-\frac12}^{\frac12}\frac{e^{2\pi i\theta'}+e^{2\pi i\theta}}{e^{2\pi i\theta'}+e^{2\pi i\theta}+q^m}d\theta'$$

$$=\begin{cases}\dfrac{e^{2\pi i\theta}}{e^{2\pi i\theta}+q^m} & \text{if } -\dfrac12+\dfrac{\alpha_m}{2\pi}<\theta<\dfrac12-\dfrac{\alpha_m}{2\pi},\\ 1 & \text{if } \dfrac12-\dfrac{\alpha_m}{2\pi}<\theta<\dfrac12\\ & \text{or } -\dfrac12<\theta<-\dfrac12+\dfrac{\alpha_m}{2\pi}\end{cases}$$

where $0<\alpha_m<\pi/2$ is defined by $\alpha_m=\cos^{-1}(q^m/2)$. Since

$$\int_{-\frac12}^{\frac12}f_m(\theta)d\theta=\frac{2\alpha_m}{\pi}=\frac{2}{\pi}\cos^{-1}\left(\frac{q^m}{2}\right),$$

we obtain Theorem 3.

Remark 1. Taking the limit $q\downarrow 1$ we have

$$\begin{aligned}\lim_{q\downarrow 1}m_q(x+y+1)&=\frac{2}{\pi}\int_0^{\frac{\pi}{3}}\theta\tan\theta d\theta=\frac{2}{\pi}\int_0^{\frac{\pi}{6}}\theta\cot\theta d\theta\\&=2\log\mathcal{S}_2\left(\frac16\right)=\frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3})\end{aligned}$$

using the double sine function $\mathcal{S}_2(x)$ (see [KK] and [O1]) and the idea of the Jackson integral.

4. Proof of theorem 4.

(1)

$$m_q(ax)=(q-1)\,\mathrm{Re}\sum_{m=1}^{\infty}\int_0^1\frac{ae^{2\pi i\theta}-1}{ae^{2\pi i\theta}-1+q^m}d\theta$$

$$= (q-1)\sum_{m=1}^{\infty}\frac{-1}{q^m-1}$$
$$= -\sum_{n=1}^{\infty}\frac{1}{[n]_q}$$
$$= l_q(0).$$

(2) From (1) we have

$$m_q(2x) = m_q(x) = -\sum_{n=1}^{\infty}\frac{1}{[n]_q},$$

and

$$m_q(2) = l_q(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[n]_q} = \sum_{m=1}^{\infty}\frac{q-1}{q^m+1} > 0.$$

(3) Since

$$m_q(4) = l_q(4) = 3(q-1)\sum_{m=1}^{\infty}\frac{1}{q^m+3}$$

and

$$m_q(2) = l_q(2) = (q-1)\sum_{m=1}^{\infty}\frac{1}{q^m+1},$$

we have

$$m_q(4) - 2m_q(2)$$
$$= (q-1)\sum_{m=1}^{\infty}\left(\frac{3}{q^m+3} - \frac{2}{q^m+1}\right) > 0.$$

Remark 2. The above (1) gives also a counter example to the expectation "$m_q(f) \geq 0$ for each f of $\mathbf{Z}$-coefficients"; the corresponding fact for $m(f)$ is a characteristic property of the Mahler measure (see [EW]).

5. Variations. It is natural to generalize $m_q(f)$ for $q \in \mathbf{C}$. We notice three cases:

(1) $|q| > 1$: The previous definitions of $l_q(x)$ and $m_q(f)$ are applicable, and results are similar.

(2) $0 < |q| < 1$: Take

$$l_q(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{[n]_q}$$

in $|x-1| < 1$ with $[n]_q = (1-q^n)/(1-q)$. It has an analytic continuation to all $x \in \mathbf{C}$ via

$$l_q(x) = (1-q)\sum_{m=0}^{\infty}\frac{x-1}{x-1+q^{-m}}.$$

Using this q-logarithm, the q-Mahler measure $m_q(f)$ is defined by the same formula as above. Moreover the calculations are quite similar to the case of $|q| > 1$.

(3) $q = 0$ ("crystal Mahler measure"): Setting

$$l_0(x) = 1 - \frac{1}{x}$$

(this being obtained as the limit $q \to 0$ in (2)), we have the crystal Mahler measure ("crystal" means "$q = 0$")

$$m_0(f)$$
$$= 1 - \mathrm{Re}\int_0^1 \cdots \int_0^1 f(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})^{-1} d\theta_1 \cdots d\theta_n.$$

For example $m_0(x+1) = 1/2$ and $m_0(x+y+1) = 2/3$. We notice the following result

$$m_q\left(\frac{x+1}{y+1}z+1\right) = \frac{4}{\pi^2}\sum_{\substack{n\geq 1\\ \text{odd}}}\frac{1}{[n]_q n^2}$$

valid for $|q| > 1$ and $|q| < 1$, which is a q-deformation of Smyth's result

$$m\left(\frac{x+1}{y+1}z+1\right) = \frac{4}{\pi^2}\sum_{\substack{n\geq 1\\ \text{odd}}}\frac{1}{n^3} = \frac{7\zeta(3)}{2\pi^2}.$$

In particular

$$m_0\left(\frac{x+1}{y+1}z+1\right) = \frac{1}{2}.$$

We refer [O2] for further examples and calculations.

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