Characterization of totally η-umbilic real hypersurfaces in nonflat complex space forms by some inequality

By Takehiro ITOH*) and Sadahiro MAEDA**)

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Abstract: In this paper we characterize all totally η-umbilic real hypersurfaces M's in complex projective or complex hyperbolic spaces by using an inequality related to the shape operator A of M.

Key words: Nonflat complex space forms; totally η-umbilic real hypersurfaces; shape operators.

1. Introduction. In an n-dimensional nonflat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c, which is either a complex projective space $\mathbf{C}P^n(c)$ or a complex hyperbolic space $\mathbf{C}H^n(c)$, there does not exist a totally umbilic real hypersurface M^{2n-1}.

However, there exist real hypersurfaces M^{2n-1} which are so-called totally η-umbilic real hypersurfaces in $\widetilde{M}_n(c)$, $c \neq 0$. A real hypersurface M of $\widetilde{M}_n(c)$ ($n \geqq 2$) (with standard Riemannian metric $\langle\ ,\ \rangle$) is called *totally η-umbilic*, if its shape operator A is of the form $AX = \alpha X$ for each vector X on M *which is orthogonal to the characteristic vector ξ of* M, where α is a smooth function on M. This definition can be rewritten easily as: $AX = \alpha X + \beta\eta(X)\xi$ for each $X \in TM$, where α, β are smooth functions on M and $\eta(X) = \langle X, \xi\rangle$. It is known that these two functions α and β are automatically constant.

The main purpose of this paper is to give a characterization of all totally η-umbilic real hypersurfaces M's of a nonflat complex space form $\widetilde{M}_n(c)$ by using an inequality related to the shape operator A of M, that is, we will prove the following

Theorem. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ ($n \geqq 2$). Then the following inequality holds*:

$$(\operatorname{trace} A - \langle A\xi, \xi\rangle)^2 \leqq 2(n-1)(\operatorname{trace} A^2 - \|A\xi\|^2),$$

where A is the shape operator of M in the ambient space $\widetilde{M}_n(c)$.

Moreover, the equality holds on M if and only if M is totally η-umbilic in $\widetilde{M}_n(c)$.

In the last section we pose an open problem on real hypersurfaces in a complex projective space.

2. Basic results on totally η-umbilic real hypersurfaces. We shall review some basic results on totally η-umbilic real hypersurfaces. Let M^{2n-1} be an orientable real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ ($n \geqq 2$) and $\mathcal{N}$ a unit normal vector field on M in $\widetilde{M}_n(c)$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by

(2.1)
$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX,$$

for vector fields X and Y tangent to M, where $\langle\ ,\ \rangle$ denotes the Riemannian metric on M induced from the standard metric on $\widetilde{M}_n(c)$, and A is the shape operator of M in $\widetilde{M}_n(c)$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of $\widetilde{M}_n(c)$. The characteristic vector field ξ of M is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \text{and}$$
$$\langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where I denotes the identity mapping of the tangent bundle TM of M. It follows from the equalities (2.1) that

(2.2)
$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi \quad \text{and} \quad \nabla_X \xi = \phi AX.$$

Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively.

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*) Faculty of Education, Shinshu University, 6-r0, Nisinagano, Nagano 380-8544.

**) Department of Mathematics, Shimane University, 1060, Nishi-Kawatsucho, Matsue, Shimane 690-8504.

We recall the classification theorem of totally η-umbilic real hypersurfaces of $\widetilde{M}_n(c)$, $c \neq 0$ ([NR]):

Theorem A. *Let M^{2n-1} be a totally η-umbilic real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ (with shape operator $A = \alpha I + \beta\eta \otimes \xi$). Then M is locally congruent to one of the following*:

(P) *geodesic spheres of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbf{C}P^n(c)$, where $\alpha = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$ and $\beta = -1/\alpha$,*

(H) i) *horospheres in $\mathbf{C}H^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$*

ii) *geodesic spheres of radius r $(0 < r < \infty)$ in $\mathbf{C}H^n(c)$, where $\alpha = (\sqrt{c}/2)\coth(\sqrt{c}r/2)$ and $\beta = 1/\alpha$,*

iii) *tubes of radius r $(0 < r < \infty)$ around totally geodesic complex hyperplane $\mathbf{C}H^{n-1}(c)$ in $\mathbf{C}H^n(c)$, where $\alpha = (\sqrt{c}/2)\tanh(\sqrt{c}r/2)$ and $\beta = 1/\alpha$.*

It is known that every totally η-umbilic hypersurface M satisfies that the structure tensor ϕ and the shape operator A of M in $\widetilde{M}_n(c)$ are commutative: $\phi A = A\phi$.

We here review the definition of circles in Riemannian geometry. A unit speed curve $\gamma = \gamma(s)$ in a Riemannian manifold M is called a *circle* if there exist a field of unit vectors $Y = Y(s)$ along the curve and a constant $\kappa\,(\geqq 0)$ which satisfy the differential equations: $\nabla_{\dot\gamma}\dot\gamma = \kappa Y$ and $\nabla_{\dot\gamma}Y = -\kappa\dot\gamma$, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The constant κ is called the curvature of the circle. A circle with zero curvature is nothing but a geodesic.

It is well-known that a hypersurface M^n in Euclidean space $\mathbf{R}^{n+1}$ is locally a standard sphere if and only if all geodesics of M are circles of positive curvature in $\mathbf{R}^{n+1}$. However, there exist no real hypersurfaces all of whose geodesics are circles in a nonflat complex space form $\widetilde{M}_n(c)$. This comes from the fact that a nonflat complex space form does not admit a totally umbilic real hypersurface.

Paying attention to the extrinsic shape of geodesics on totally η-umbilic real hypersurfaces in $\widetilde{M}_n(c)$ $(c \neq 0)$, we obtain the following which is a characterization of these hypersurfaces ([MO]):

Proposition B. *Let M^{2n-1} be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then M is locally congruent to a totally η-umbilic real hypersurface if and only if every geodesic γ on M whose initial vector $\dot\gamma(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ of M is a circle of positive curvature in the ambient space $\widetilde{M}_n(c)$.*

The following result shows that Proposition B is no longer true if we replace "a circle of positive curvature" by "a circle" ([AKM]):

Theorem C. *Let M^{2n-1} be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then M is locally congruent to a totally η-umbilic real hypersurface or a ruled real hypersurface if and only if every geodesic γ on M whose initial vector $\dot\gamma(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ of M is a circle in the ambient space $\widetilde{M}_n(c)$.*

It is well-known that the characteristic vector ξ of each ruled real hypersurface is not principal (for details, see [NR]).

3. Characterizations of totally $\boldsymbol{\eta}$-umbilic real hypersurfaces. Our tool in this section is the first equality in (2.2). We first prove the following:

Proposition 1. *The structure tensor ϕ of each real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(c \neq 0)$ is not parallel, namely $\nabla\phi$ does not vanish identically on M.*

Proof. Suppose that $\nabla\phi \equiv 0$ on M. Then it follows from the first equality in (2.2) that

$$\eta(Y)AX - \langle AX, Y\rangle\xi = 0 \quad \text{for } \forall X, Y \in TM. \tag{3.1}$$

Putting $X = Y = \xi$ in (3.1), we can see that ξ is principal. Next for each $X\,(\neq 0)$ orthogonal to ξ with $AX = rX$, putting $Y = \xi$ in (3.1), we get $r = 0$. So our real hypersurfce is totally η-umbilic with $\alpha = 0$ in the ambient space $\widetilde{M}_n(c)$, which is a contradiction (see Theorem A). □

The following is a characterization of a totally η-umbilic real hypersurface in terms of the covariant derivative of its structure tensor ϕ:

Proposition 2. *Let M^{2n-1} be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following are equivalent*:

(1) *M is totally η-umbilic in the ambient space $\widetilde{M}_n(c)$.*

(2) *The structure tensor ϕ of M satisfies*

$$(\nabla_X\phi)Y = k(\eta(Y)X - \langle X, Y\rangle\xi) \quad \text{for } \forall X, Y \in TM, \tag{3.2}$$

where k is a nonzero constant.

Proof. (1) $\Longrightarrow$ (2): Suppose that $AX = \alpha X + \beta\eta(X)\xi$ for $\forall X, Y \in TM$. Then it follows from the first equality in (2.2) that $(\nabla_X\phi)Y = \alpha(\eta(Y)X - \langle X,Y\rangle\xi)$ for $\forall X, Y \in TM$.

(2) $\Longrightarrow$ (1): In view of the first equality in (2.2) and (3.2) we find that

$$\eta(Y)AX - \langle AX,Y\rangle\xi = k(\eta(Y)X - \langle X,Y\rangle\xi) \quad \text{for } \forall X,Y \in TM. \tag{3.3}$$

Putting $X = Y = \xi$ in (3.3), we can see that ξ is principal. Next for each $X\,(\neq 0)$ orthogonal to ξ with $AX = rX$, putting $Y = \xi$ in (3.3), we get $r = k$. So we obtain the desirable conclusion. □

We are now in a position to prove our theorem:

Proof of main theorem. Making use of (3.2), we define the following tensor T on M as:

$$T(X,Y) = (\nabla_X\phi)Y - k(\eta(Y)X - \langle X,Y\rangle\xi) \quad \text{for } \forall X,Y \in TM.$$

Calculating the length of T, we obtain the following inequalty

$$\|T\|^2 = 2\,\mathrm{trace}\,A^2 + 4(n-1)k^2 - 2\|A\xi\|^2 + 4k\langle A\xi,\xi\rangle - 4k\cdot\mathrm{trace}\,A \geqq 0,$$

so that for each k we see that

$$2(n-1)k^2 + 2(\langle A\xi,\xi\rangle - \mathrm{trace}\,A)k + \mathrm{trace}\,A^2 - \|A\xi\|^2 \geqq 0. \tag{3.4}$$

This tells us that the discriminant D of the quadratic function (3.4) is nonpositive. Thus we obtain the conclusion. □

4. Open problem. Totally η-umbilic real hypersurfaces are typical examples of *homogeneous* real hypersurfaces, namely they are given as orbits under subgroups of the isometry group $I(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$. The classification problem of homogeneous real hypersurfaces in $\mathbf{C}H^n(c)$ is still open. However, in $\mathbf{C}P^n(c)$ the classification problem of such hypersurfaces is completely solved. Here, without loss of generality we put $c = 4$. We recall the following ([NR]):

Theorem D. *Let M be a homogeneous real hypersurface of $\mathbf{C}P^n(4)$. Then M is a tube of radius r over the following Kähler submanifolds*:

(A_1) *hyperplane $\mathbf{C}P^{n-1}(4)$, where $0 < r < \pi/2$,*

(A_2) *totally geodesic $\mathbf{C}P^k(4)$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$,*

(B) *complex hyperquadric Q_{n-1}, where $0 < r < \pi/4$,*

(C) *$\mathbf{C}P^1(4) \times \mathbf{C}P^{\frac{n-1}{2}}(4)$, where $0 < r < \pi/4$ and $n\,(\geqq 5)$ is odd,*

(D) *complex Grassmann $\mathbf{C}G_{2,5}$, where $0 < r < \pi/4$ and $n = 9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

The numbers of distinct principal curvatures of these homogeneous real hypersurfaces are 2, 3, 3, 5, 5, 5, respectively. Note that a geodesic sphere of radius r in $\mathbf{C}P^n(4)$ is congruent to a tube of radius $(\pi/2)-r$ over hyperplane $\mathbf{C}P^{n-1}(4)$ in $\mathbf{C}P^n(4)$, where $0 < r < \pi/2$.

Motivated by Propositions 1 and 2, we are interested in $\|\nabla\phi\|$ for each minimal homogeneous real hypersurface in $\mathbf{C}P^n(4)$. Direct computation yields the following:

Proposition 3. *Let M be a minimal homogeneous real hypersurface (which is a tube of radius r) of $\mathbf{C}P^n(4)$. Then the radius r and the norm of the covariant derivative of the structure tensor ϕ on M are as follows*:

(A_1) $\cot r = \sqrt{2n-1}$ *and* $\|\nabla\phi\|^2 = \frac{4(n-1)}{2n-1}$,

(A_2) $\cot r = \sqrt{\frac{2k+1}{2n-2k-1}}$ *and* $\|\nabla\phi\|^2 = 4\{\frac{(n-1-k)(2k+1)}{2n-1-2k} + \frac{k(2n-1-2k)}{2k+1}\}$,

(B) $\cot r = \sqrt{n} + \sqrt{n-1}$ *and* $\|\nabla\phi\|^2 = 4(n+1)$,

(C) $\cot r = \frac{\sqrt{n}+\sqrt{2}}{\sqrt{n-2}}$ *and* $\|\nabla\phi\|^2 = 12n - 4 - \frac{16}{n-2}$,

(D) $\cot r = \sqrt{5}$ *and* $\|\nabla\phi\|^2 = \frac{488}{5}(= 97.6)$,

(E) $\cot r = \frac{\sqrt{15}+\sqrt{6}}{3}$ *and* $\|\nabla\phi\|^2 = \frac{512}{3}(= 170.6\cdots)$.

Proposition 3 tells us that $\|\nabla\phi\|^2$ takes the minimal value in case of type A_1 and takes the maximum value in case of type C. In this context we pose the following problem:

Problem. Let M be a compact orientable minimal real hypersurface of $\mathbf{C}P^n(4)$. If the covariant derivative of the structure tensor ϕ satisfies $\|\nabla\phi\|^2 \leqq 4(n-1)/(2n-1)$ on M, is M congruent to a geodesic sphere in $\mathbf{C}P^n(4)$?

We emphasize that $\|\nabla\phi\|$ is a natural invariant for each real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. In general, for any real hypersurface M the following holds: $\|\nabla\phi\|^2 = 2(\mathrm{trace}\,A^2 - \|A\xi\|^2)$. The following proposition is worth mentioning.

Proposition 4. *Let M be a minimal homogeneous real hypersurface of $\mathbf{C}P^n(4)$. Then* trace A^2 *of M is described as follows*:

(1) *If M is of type* A_1 *or* A_2, *then* $\mathrm{trace}\,A^2 = 2n-2$.

(2) *If* M *is of type* B, C, D *or* E, *then* trace $A^2 = 6n - 2$.

Proposition 4 shows that we cannot distinguish minimal homogeneous real hypersurfaces of type A_1 and A_2, and also minimal ones of type B, C, D and E in terms of trace A^2. The following is well-known ([NR]):

Theorem E. *Let* M *be a compact orientable minimal real hypersurface of* $\mathbf{C}P^n(4)$. *Suppose that the shape operator* A *of* M *in* $\mathbf{C}P^n(4)$ *satisfies* trace $A^2 \leqq 2n - 2$ *on* M. *Then* trace $A^2 \equiv 2n - 2$ *and* M *is congruent to one of minimal homogeneous real hypersurfaces of type* A_1 *and* A_2.

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