Impulsive cellular neural networks and almost periodicity

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Abstract: In the following article we present the results on the existence of almost periodic solutions for impulsive neural networks. We also formulate several results on exponential stability of these equations.

Key words: Almost periodic solutions; impulsive neural networks.

1. Introduction. The mathematical models of many problems and phenomena in the real world can be described with impulsive differential equations of the form

$$\begin{cases} \dot{x}(t) = F(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = I_k(x(t)), & t = \tau_k,\ k \in \mathbf{Z}, \end{cases}$$

where t belongs to the interval $J \subset \mathbf{R}$, $F : J \times \mathbf{R}^n \to \mathbf{R}^n$, the sequence $\{\tau_k\}$ has no finite accumulation point, $\Delta x(t) = x(t+0) - x(t-0)$, $I_k : R^n \to \mathbf{R}^n$.

The theory of impulsive differential equations goes back to the works of Mil'man and Myshkis [13]. In the recent years impulsive differential equations have been intensively researched (see the monographs of Samoilenko and Perestyuk [2] and Lakshmikantham et al. [12]). Recently, some qualitative properties (oscillation, asymptotic behavior and stability) are investigated by several authors (see [9–10]).

In this paper, we investigate the existence and attractivity of almost periodic solutions for impulsive cellular neural networks. It is well know that the neural networks have successful applications in many fields such as optimization, associative memory, signal and image procesing. Many authors have paid much attention to research on the theory and aplication of the cellular neural networks.

The main results related to the study of the existence almost periodic solutions for system with impulses effects have been obtained in [3–8].

2. Preliminary notes. Let in n-dimensional Euclidean space $\mathbf{R}^n$ with elements $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$ the norm is defined by $|x| = \max_i\{|x_i|\}$, $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0, +\infty)$, Ω be a domain in $\mathbf{R}^n$, $\Omega \neq \emptyset$.

By B, $B = \{\{\tau_k\}_{k=-\infty}^{\infty} : \tau_k \in \mathbf{R},\ \tau_k < \tau_{k+1},\ k \in \mathbf{Z},\ \lim_{k\to\pm\infty} \tau_k = \pm\infty\}$ we denote the set of all sequences unbounded and strictly increasing.

We shall investigate the problem of existence of almost periodic solutions of the system of impulsive cellular neural networks in the form

(1)
$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} \alpha_{ij}(t) f_j(x_j(t)) \\ \quad + \sum_{j=1}^{n} \beta_{ij}(t) f_j\Big(\mu_j \int_0^\infty k_{ij}(u)x_j(t-u)du\Big) + \gamma_i(t), \\ \qquad\qquad t \neq \tau_k,\ i = 1, 2, \ldots, n, \\ \Delta x(t) = A_k x(t) + I_k(x(t)) + \gamma_k, \\ \qquad\qquad t = \tau_k,\ k \in \mathbf{Z}, \end{cases}$$

where

(i) $t \in \mathbf{R}$, $a_{ij}(t)$, $\alpha_{ij}(t)$, $\beta_{ij}(t) \in C(\mathbf{R}, \mathbf{R})$, $f_j(t) \in C(\mathbf{R}, \mathbf{R})$, $\mu_j \in \mathbf{R}_+$, $k_{ij}(t) \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\gamma_i(t) \in C(\mathbf{R}, \mathbf{R})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$;

(ii) $A_k \in \mathbf{R}^{n\times n}$, $I_k(x) \in C(\Omega, \mathbf{R}^n)$, $\gamma_k \in \mathbf{R}^n$, $\{\tau_k\} \in B$, $k \in \mathbf{Z}$.

Let $PC(J, \mathbf{R}^n)$, $J \subset \mathbf{R}$ is the space of all piecewise continuous functions $x : J \to \mathbf{R}^n$ with points of discontinuity of first kind τ_k in which it is left continuous, i.e. the following relations hold

$$x(\tau_k - 0) = x(\tau_k),\ x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k),\ k \in \mathbf{Z}.$$

Recall [9] that the solution $x(t)$ of (2) is from $PC(J, \mathbf{R}^n)$.

The initial condition associated with (1) is of the form

(2) $$x(t) = \phi_0(t), \quad t \in \mathbf{R},$$

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where $\phi_0(t) \in PC(\mathbf{R}, \mathbf{R}^n)$ is almost periodic function with points of discontinuity of first kind τ_k, $k \in \mathbf{Z}$.

Since the solutions of (1), (2) are piecewise functions we adopt from [1] the following definitions for almost periodicity.

Definition 1 [2]**.** The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ is said to be *uniformly almost periodic* if for arbitrary $\varepsilon > 0$ there exists relatively dense set of ε-almost periods common for any sequences.

Definition 2 [2]**.** The function $\varphi \in PC(\mathbf{R}, \mathbf{R}^n)$ is said to be *almost periodic*, if:

a) the set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in Z$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ and it is uniformly almost periodic.

b) for any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$.

c) for any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|\varphi(t+\tau) - \varphi(t)| < \varepsilon$ for all $t \in R$ satisfying the condition $|t - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$.

The elements of T are called *ε-almost periods of* $\varphi(t)$.

Together with the system (1) we consider the linear system

$$(3)\qquad \begin{cases} \dot{x}(t) = A(t)x(t), & t \neq \tau_k, \\ \Delta x(t) = A_k x(t), & t = \tau_k,\ k \in \mathbf{Z}, \end{cases}$$

where $t \in \mathbf{R}$, $A(t) = (a_{ij}(t))$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$.

Introduce the following conditions:

H1. $A(t) \in C(\mathbf{R}, \mathbf{R}^n)$ and is almost periodic in the sense of Bohr.

H2. $\det(E + A_k) \neq 0$ and the sequence $\{A_k\}$, $k \in \mathbf{Z}$ is almost periodic, $E \in \mathbf{R}^{n\times n}$.

H3. The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta > 0$.

Recall [9] that if $U_k(t, s,)$ is the Cauchy matrix for the system

$$\dot{x}(t)dt = A(t)x(t),\quad \tau_{k-1} < t \leq \tau_k,\ \{\tau_k\} \in B$$

then the Cauchy matrix for the system (3) is in the form

$$W(t,s) = \begin{cases} U_k(t,s),\ \tau_{k-1} < s \leq t \leq \tau_k, \\ U_{k+1}(t, \tau_k + 0)(E + A_k)U_k(t,s), \\ \qquad \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}, \\ U_{k+1}(t, \tau_k + 0)(E + A_k)U_k(\tau_k, \tau_k + 0) \\ \qquad \cdots (E + A_i)U_i(\tau_i, s), \\ \tau_{i-1} < s \leq \tau_i < \tau_k < t \leq \tau_{k+1} \end{cases}$$

and the solutions of (3) are writen in the form

$$x(t; t_0, x_0) = W(t, t_0)x_0.$$

Lemma 1 [2]**.** *Let the following conditions be fulfilled*:

1. *Conditions* H1–H3 *are fulfilled.*
2. *For the Cauchy matrix $W(t,s)$ of the system* (3) *there exist positive constants K and λ such that*

$$|W(t,s)| \leq Ke^{-\lambda(t-s)},\quad t \geq s,\ t, s \in \mathbf{R}.$$

Then for any $\varepsilon > 0$, $t \in \mathbf{R}$, $s \in \mathbf{R}$, $t \geq s$, $|t - \tau_k| > \varepsilon$, $|s - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$ there exists a relatively dense set T of ε-almost periods of the matrix $A(t)$ and a positive constant Γ such that for $\tau \in T$ it follows

$$|W(t + \tau, s + \tau) - W(t,s)| \leq \varepsilon\Gamma e^{-(\lambda/2)(t-s)}.$$

Introduce the following conditions:

H4. The functions $\alpha_{ij}(t)$ are almost periodic in the sense of Bohr, and

$$0 < \sup_{t\in\mathbf{R}} |\alpha_{ij}(t)| = \overline{\alpha}_{ij} < \infty.$$

H5. The functions $\beta_{ij}(t)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ are almost periodic in the sense of Bohr, and

$$0 < \sup_{t\in\mathbf{R}} |\beta_{ij}(t)| = \overline{\beta}_{ij} < \infty.$$

H6. The functions $f_j(t)$ are almost periodic in the sense of Bohr,

$$0 < \sup_{t\in\mathbf{R}} |f_j(t)| < \infty,\quad f_j(0) = 0,$$

and there exists $L_1 > 0$ such that for $t, s \in \mathbf{R}$

$$\max_j |f_j(t) - f_j(s)| < L_1|t - s|,\quad j = 1, 2, \ldots, n.$$

H7. The functions $k_{ij}(t)$ satisfies

$$\int_0^\infty k_{ij}(s)ds = 1,\quad \int_0^\infty s k_{ij}(s)ds < \infty,$$
$$i, j = 1, 2, \ldots, n.$$

H8. The functions $\gamma_i(t)$, $i = 1, 2, \ldots, n$ are almost periodic in the sense of Bohr, $\{\gamma_k\}_{k\in\mathbf{Z}}$ is almost periodic sequence and there exists $C_0 > 0$ such that

$$\max\Big\{\max_i |\gamma_i(t)|, \max_k |\gamma_k|\Big\} \le C_0.$$

H9. The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in \Omega$ and there exists $L_2 > 0$ such that

$$|I_k(x) - I_k(y)| \le L_2|x - y|$$

for $k \in \mathbf{Z}$, $x, y \in \Omega$.

Lemma 2 [2]. *Let the conditions* H1–H6, H8 *be fulfilled. Then for each* $\varepsilon > 0$ *there exist* ε_1, $0 < \varepsilon_1 < \varepsilon$ *and relatively dense sets* T *of real numbers and* Q *of whole numbers, such that the following relations are fulfilled*:

(a) $|A(t+\tau) - A(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$;
(b) $|\alpha_{ij}(t+\tau) - \alpha_{ij}(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t-\tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $i, j = 1, 2, \dots, n$;
(c) $|\beta_{ij}(t+\tau) - \beta_{ij}(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t-\tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $i, j = 1, 2, \dots, n$;
(d) $|f_j(t+\tau) - f_j(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t-\tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $j = 1, 2, \dots, n$;
(e) $|A_{k+q} - A_k| < \varepsilon$, $t \in \mathbf{R}$, $q \in Q$, $k \in \mathbf{Z}$;
(f) $|\gamma_j(t+\tau) - \gamma_j(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t-\tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $j = 1, 2, \dots, n$;
(g) $|\gamma_{k+q} - \gamma_k| < \varepsilon$, $t \in \mathbf{R}$, $q \in Q$, $k \in \mathbf{Z}$;
(h) $|\overline{\tau}_k^q - \tau| < \varepsilon_1$, $q \in Q$, $\tau \in T$, $k \in \mathbf{Z}$.

Lemma 3 [2]. *Let the set of sequences* $\{\tau_k^j\}$ *be uniformly almost periodic. Then for each* $p > 0$ *there exists a positive integer* N *such that on each interval of length* p *no more than* N *elements of the sequence* $\{\tau_k\}$, *i.e.*,

$$i(s, t) \le N(t - s) + N,$$

where $i(s, t)$ *is the number of points* τ_k *in the interval* (s, t).

3. Main results

Theorem 1. *Let the following conditions be fulfilled*:

1. *Conditions* H1–H8 *are fulfilled.*
2. *The number*

$$r = K\Big\{\max_i \lambda^{-1} L_1 \sum_{j=1}^{n} (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j) + \frac{L_2}{1 - e^{-\lambda}}\Big\} < 1.$$

Then:

1. *There exists unique almost periodic solution* $x(t)$ *of* (1).
2. *If the following inequalities hold*

$$1 + KL_2 < e,$$

$$\lambda - KL_1 \max_i \sum_{j=1}^{n} (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j) - N\ln(1 + KL_2) > 0$$

then the solution $x(t)$ *is exponentially stable.*

Proof of assertion 1. We denote with D, $D \subset PC(\mathbf{R}, \mathbf{R}^n)$ the set of all almost periodic functions $\varphi(t)$ satisfying the inequality $||\varphi|| < \overline{K}$, $||\varphi|| = \sup_{t\in\mathbf{R}} |\varphi(t)|$, $\overline{K} = KC_0\big(\frac{1}{\lambda} + \frac{1}{1-e^{-\lambda}}\big)$.

Set

$$G(t, x) = \operatorname{col}\{G_1(t, x), G_2(t, x), \dots, G_n(t, x)\},$$
$$\gamma(t) = \operatorname{col}(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)),$$

where

$$G_i(t, x) = \sum_{j=1}^{n} \alpha_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t) f_j\Big(\mu_j \int_0^\infty k_{ij}(u) x_j(t-u)\,du\Big), \quad i = 1, 2, \dots, n.$$

Define in D an operator S,

$$S\varphi = \int_{-\infty}^{t} W(t, s)[G(s, \varphi(s)) + \gamma(s)]\,ds + \sum_{\tau_k < t} W(t, \tau_k)[I_k(\varphi(\tau_k)) + \gamma_k], \tag{4}$$

and subset D^*, $D^* \subset D$,

$$D^* = \Big\{\varphi \in D : ||\varphi - \varphi_0|| \le \frac{r\overline{K}}{1 - r}\Big\},$$

where

$$\varphi_0 = \int_{-\infty}^{t} W(t, s)\gamma(s)\,ds + \sum_{t_k < t} W(t, \tau_k)\gamma_k.$$

We have

$$\begin{aligned}
||\varphi_0|| &= \sup_{t\in\mathbf{R}}\Big\{\max_i\Big(\int_{-\infty}^{t} |W(t, s)||\gamma_i(s)|\,ds\Big) + \sum_{\tau_k < t} |W(t, \tau_k)||\gamma_k|\Big\} \\
&\le \sup_{t\in\mathbf{R}}\Big\{\max_i\Big(\int_{-\infty}^{t} Ke^{-\lambda(t-s)}|\gamma_i(s)|\,ds\Big) + \sum_{\tau_k < t} Ke^{-\lambda(t-\tau_k)}|\gamma_k|\Big\} \\
&\le K\Big(\frac{C_0}{\lambda} + \frac{C_0}{1 - e^{-\lambda}}\Big) = \overline{K}.
\end{aligned} \tag{5}$$

Then for arbitrary $\varphi \in D^*$ from (4) and (5) we have

$$||\varphi|| \le ||\varphi - \varphi_0|| + ||\varphi_0|| \le \frac{r\overline{K}}{1-r} + \overline{K} = \frac{\overline{K}}{1-r}.$$

Now we prove that S is self-mapping from D^* to D^*.

For arbitrary $\varphi \in D^*$ it follows

$$(6) \quad ||S\varphi - \varphi_0||$$
$$= \sup_{t\in\mathbf{R}} \Big\{ \max_i \int_{-\infty}^t |W(t,s)| \Big(\sum_{j=1}^n |\alpha_{ij}(s) f_j(x_j(s))|$$
$$+ \sum_{j=1}^n |\beta_{ij}(s)| \Big| f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \varphi_j(s-u) du \Big) \Big| ds \Big)$$
$$+ \sum_{\tau_k < t} |W(t,\tau_k)| |I_k(\varphi(\tau_k))| \Big\}$$
$$\le \Big\{ \max_i \Big(\int_{-\infty}^t K e^{-\lambda(t-s)} L_1 \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j) ds \Big)$$
$$+ \sum_{\tau_k<t} K e^{-\lambda(t-\tau_k)} L_2 \Big\} ||\varphi||$$
$$\le K \Big\{ \max_i \lambda^{-1} L_1 \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j) + \frac{L_2}{1-e^{-\lambda}} \Big\} ||\varphi||$$
$$= r||\varphi|| \le \frac{r\overline{K}}{1-r}.$$

Let $\tau \in T$, $q \in Q$ where the sets T and Q are determined in Lemma 2.

Then

$$(7) \quad ||S\varphi(t+\tau) - S\varphi(t)||$$
$$\le \sup_{t\in\mathbf{R}} \Big\{ \max_i \Big(\int_{-\infty}^t |W(t+\tau, s+\tau) - W(t,s)|$$
$$\times \Big| \sum_{j=1}^n \alpha_{ij}(s) f_j(\varphi_j(s+\tau))$$
$$+ \sum_{j=1}^n \beta_{ij}(s+\tau) f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \varphi_j(s+\tau-u) du \Big) \Big| ds$$
$$+ \int_{-\infty}^t |W(t,s)| \Big| \sum_{j=1}^n \alpha_{ij}(s) f_j(\varphi_j(s+\tau))$$
$$+ \sum_{j=1}^n \beta_{ij}(s+\tau) f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \varphi_j(s+\tau-u) du \Big)$$
$$- \sum_{j=1}^n \alpha_{ij}(s) f_j(\varphi_j(s))$$
$$- \sum_{j=1}^n \beta_{ij}(s) f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \varphi_j(s-u) du \Big) \Big| ds \Big)$$
$$+ \sum_{\tau_k<t} |W(t+\tau, \tau_{k+q}) - W(t,\tau_k)| |I_{k+q}(\varphi(\tau_{k+q}))|$$
$$+ \sum_{\tau_k<t} |W(t,\tau_k)| |I_{k+q}(\varphi(\tau_{k+q})) - I_k(\varphi(\tau_k))| \Big\}$$
$$\le \varepsilon C_1$$

where

$$C_1 = \frac{L_1}{\lambda} \Big(\max_i \Big(\sum_{j=1}^n (2\Gamma + K) \overline{\beta}_{ij} \mu_j \Big) + K \Big) + \frac{L_2 \Gamma N}{1 - e^{-\lambda}}.$$

From (6) and (7) we obtain that $S\varphi \in D^*$.

Let $\varphi \in D^*$, $\psi \in D^*$. We get

$$(8) \quad ||S\varphi - S\psi||$$
$$\le \sup_{t\in\mathbf{R}} \Big\{ \max_i \Big(\int_{-\infty}^t |W(t,s)|$$
$$\Big[\sum_{j=1}^n |\alpha_{ij}(s)| |f_j(\varphi_j(s)) - f_j(\psi_j(s))|$$
$$+ \sum_{j=1}^n |\beta_{ij}(s)| \Big| f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \varphi_j(s-u) du \Big)$$
$$- f_j \Big(\mu_j \int_0^\infty k_{ij}(u) \psi_j(s-u) du \Big) \Big| \Big] ds \Big)$$
$$+ \sum_{\tau_k<t} |W(t,\tau_k)| |I_k(\varphi(\tau_k)) - I_k(\psi(\tau_k))| \Big\}$$
$$\le K \Big(\max_i \Big(\lambda^{-1} L_1 \sum_{j=1}^n \overline{\beta}_{ij} \mu_j \Big) + \frac{L_2}{1-e^{-\lambda}} \Big) ||\varphi - \psi||$$
$$= r||\varphi - \psi||.$$

Then from (8)it follows that S is contracting operator in D^*. So there exists unique almost periodic solution of (1). □

Proof of assertion 2. Let $y(t)$ be arbitrary solution of (1) with initial condition $y(t_0+0, t_0, \varpi_0) = \varpi_0$, $\varpi_0 \in PC(t_0)$. Then from (3) we obtain

$$y(t) - x(t) = W(t,t_0)(\varpi_0 - \varphi_0)$$
$$+ \int_{t_0}^t W(t,s)[G(s,y(s)) - G(s,x(s))] ds$$
$$+ \sum_{t_0<\tau_k<t} W(t,\tau_k)[I_k(y(\tau_k)) - I_k(x(\tau_k))].$$

Then

$$|y(t) - x(t)| \le K e^{-\lambda(t-t_0)} |\varpi_0 - \varphi_0|$$
$$+ \max_i \Big(\int_{t_0}^t K e^{-\lambda(t-s)} L_1$$
$$\times \sum_{j=1}^n (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j) |y_i(s) - x_i(s)| ds \Big)$$
$$+ \sum_{t_0<\tau_k<t} K e^{-\lambda(t-\tau_k)} L_2 |y(\tau_k) - x(\tau_k)|.$$

Set $u(t) = |y(t) - x(t)|e^{\lambda t}$ and from Gronwall-Bellman's lemma [2] we have

$$\begin{aligned}
&|y(t) - x(t)| \\
&\le K|\varpi_0 - \varphi_0|(1 + KL_2)^{i(t_0,t)} \\
&\times \exp\Big(-\lambda + KL_1 \max_i \sum_{j=1}^{n} (\overline{\alpha}_{ij} + \overline{\beta}_{ij}\mu_j)\Big)(t - t_0).
\end{aligned}$$

□

Thus Theorem 1 is complete.

We note that the main inequalities which are used in proof of Theorem 1 are connect with the properies of the matrix $W(t, s)$ for a system (3). Now we will consider special case in which these properties are accomplished.

Example 1. Now consider the classical model of impulsive Hopfield neural networks

(9)

$$\begin{cases}
\dot{x}_i(t) = -a_i(t)x_i(t) + \displaystyle\sum_{j=1}^{n} \alpha_{ij} f_j(x_j(t)) \\
\quad + \displaystyle\sum_{j=1}^{n} \beta_{ij} f_j\Big(\mu_j \int_0^\infty k_{ij}(u)x_j(t-u)du\Big) + \gamma_i(t), \\
\qquad\qquad t \ne \tau_k, \quad i = 1, 2, \ldots, n, \\
\Delta x(t) = A_k x(t) + I_k(x(t)) + \gamma_k, \quad t = \tau_k, \ k \in \mathbf{Z},
\end{cases}$$

where

(i) $t \in \mathbf{R}$, $a_i(t) \in C(\mathbf{R}, \mathbf{R})$, α_{ij}, $\beta_{ij} \in \mathbf{R}$, $f_j(t) \in C(\mathbf{R}, \mathbf{R})$, $\mu_j \in \mathbf{R}_+$, $k_{ij}(t) \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\gamma_i(t) \in C(\mathbf{R}, \mathbf{R})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$;

(ii) $A_k \in \mathbf{R}^{n\times n}$, $I_k(x) \in C(\Omega, \mathbf{R}^n)$, $\gamma_k \in \mathbf{R}^n$, $\{\tau_k\} \in B$, $k \in \mathbf{Z}$.

Lemma 4. *Let the following conditions be fulfilled*:

1. *For the matrix* $A(t) = \mathrm{diag}[-a_1(t), -a_2(t), \ldots, -a_1(t)]$ *it follows that* $a_i(t)$ $i = 1, 2, \ldots, n$ *is almost periodic function in the sense of Bohr and*

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} a_i(t)dt > 0, \quad i = 1, 2, \ldots, n.$$

2. *The conditions* H2, H3 *are fulfilled.*

Then for the Cauchy's matrix $W(t, s)$ *it follows*

$$|W(t, s)| \le Ke^{-\lambda(t-s)},$$

where $t \in R$, $s \in R$ $t \ge s$, K, λ *are positive constants.*

Proof. The proof of Lemma 4 is analogous with the proof of Lemma 2 from [3]. □

Theorem 2. *Let the following conditions be fulfilled*:

1. *Conditions of Lemma* 4 *are fulfilled.*
2. *Conditions* H6–H9 *are fulfilled.*
3. *The number*

$$r = K\Big\{\lambda^{-1}L_1 \sum_{j=1}^{n}(\alpha_{ij} + \beta_{ij}\mu_j) + \frac{L_2}{1 - e^{-\lambda}}\Big\} < 1.$$

Then there exists unique almost periodic solution $x(t)$ *of* (9).

If the following inequalities hold

$$\begin{aligned}
&1 + KL_2 < e, \\
&\lambda - KL_1 \sum_{j=1}^{n}(\alpha_{ij} + \beta_{ij}\mu_j) - N\ln(1 + KL_2) > 0
\end{aligned}$$

then the solution $x(t)$ *is exponentially stable.*

Proof. The proof of Theorem 2 it follows from Lemma 4, the proof of Lemma 2 and the proof of Theorem 1. □

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