A classification of three dimensional regular projectively Anosov flows

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Abstract: We give a classification of C^2-regular projectively Anosov flows on closed three dimensional manifolds. More precisely, we show that if the manifold is connected then such a flow must be either an Anosov flow or represented as a finite union of $\mathbf{T}^2 \times I$-models.

Key words: Projectively Anosov system; conformally Anosov systems; bi-contact structures.

1. Introduction. In [6], Eliashberg and Thurston developed a theory of confoliations which are mixture of foliations and contact structures on a three dimensional manifold. One of the fundamental results is that any foliation on a three dimensional manifold except $\mathcal{F} = \{S^2 \times \{*\}\}$ on $S^2 \times S^1$ can be perturbed into a positive (or negative) contact structure as a plane field. They also introduced a special class of perturbations, so called linear perturbations. *A linear perturbation* of a foliation generated by a plane field ξ is a one parameter family $\{\operatorname{Ker}\alpha_t\}_{t\in(-\epsilon,\epsilon)}$ of plane fields defined by a family of 1-forms $\{\alpha_t\}$ with $\xi = \operatorname{Ker}\alpha_0$ and $(d/dt)(\alpha_t \wedge d\alpha_t) > 0$. Eliashberg and Thurston observed that if the kernel of $\beta = (d/dt)\alpha_t|_{t=0}$ is also a foliation, then $(\operatorname{Ker}(\alpha + t\beta), \operatorname{Ker}(\alpha - t\beta))$ is a pair of mutually transverse positive and negative contact structures for any $t \neq 0$. Independently, Mitsumatsu [10] also studied the same deformation for Anosov foliations and he called such a pair of contact structure *a bi-contact structure*. Mitsumatsu, and Eliashberg and Thurston observed that for any bi-contact structure (ξ, η) the line field $\xi \cap \eta$ generates a flow with a special property, which is called *a projectively Anosov* (*or simply* **P**A) *flow* (or *a conformally Anosov flow* in [6]).

Similar to an Anosov flow, a **P**A flow preserves two mutually transverse plane fields, which are called *the stable and unstable subbundles*. When these plane fields are smooth, we can define a linear deformation which gives a bi-contact structure. Unfortunately, some **P**A flows preserve no smooth plane field. In this paper, we focus only on *regular* **P**A *flows*, which admit the smooth stable and unstable subbundles. They correspond to foliations which admits a linear deformation whose derivative generates another foliation.

In [12], Noda gave a classification of regular **P**A flows on a $\mathbf{T}^2$-bundle over S^1 having an invariant torus. After that, he and Tsuboi gave a classification for some manifolds, which can be summarized as follows:

Theorem ([12–14], and [16])**.** *Any regular* **P**A *flow on a Seifert manifold or a* $\mathbf{T}^2$*-bundle over* S^1 *must be either an Anosov flow or represented as a finite union of* $\mathbf{T}^2 \times I$*-models.*

A $\mathbf{T}^2 \times I$-model is an explicitly described **P**A-flow on $\mathbf{T}^2 \times [0, 1]$. Roughly speaking, it is a flow transverse to $\mathbf{T}^2 \times \{z\}$ for any $z \in (0, 1)$ and is equivalent to a linear flow on each boundary. See [12] for details. Since Anosov flows with smooth invariant foliations are classified by Ghys [7], it completes the classification on the above manifolds. The author also approached the classification problem from another direction. In [2], it is shown that any regular **P**A flow on *any* three dimensional closed manifold without non-hyperbolic periodic orbits is equivalent to one of the flows classified above.

In [13], Noda conjectured that there are no regular **P**A flows other than the ones classified above. The aim of this paper is to show that the conjecture is true. Namely,

Main Theorem. *Any* C^2*-regular* **P**A *flow on a connected and closed three dimensional manifold must be either an Anosov flow or represented as a finite union of* $\mathbf{T}^2 \times I$ *models.*

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An immediate corollary is an affirmative answer to a conjecture posed by Mitsumatsu (Conjecture 4.3.3 in [11]).

Corollary 1.1. *Any bi-contact structure associated with a regular* **P**A *flow consists of tight contact structures.*

We give the precise definition of regular **P**A flows and a sketch of the proof of the main theorem in the next section. The detail of the proof will appear in [3].

2. A sketch of the proof.

2.1. Definitions. First of all, we recall the definition of regular **P**A flows.

Let M be a three dimensional closed manifold and $\Phi = \{\Phi^t\}_{t \in \mathbf{R}}$ a flow on M. Let $T\Phi$ denote the one dimensional subbundle of TM which is generated by the vector field associated with Φ. The differential of Φ induces a flow $\hat{D}\Phi = \{\hat{D}\Phi^t\}_{t \in \mathbf{R}}$ on $TM/T\Phi$. A pair (E^u, E^s) of continuous two dimensional subbundles of TM is called *a projectively Anosov* (*or simply* **P**A) *splitting* if

1. $E^u \cap E^s = T\Phi$,
2. both E^u and E^s are Φ-invariant, and
3. there exist two constants $C > 0$ and $\lambda \in (0,1)$ such that

$$\|\hat{D}\Phi^{-t}|_{(E^u/T\Phi)(\Phi^t(z))}\| \cdot \|\hat{D}\Phi^t|_{(E^s/T\Phi)(z)}\| \leq C\lambda^t$$

for any $t > 0$ and $z \in M$, where $\|\cdot\|$ is a norm on $TM/T\Phi$.

It is easy to see that the definition does not depend on the choice of the norm $\|\cdot\|$ and the pair (E^u, E^s) is uniquely determined if it exists. We call E^u and E^s *the unstable and the stable subbundles* associated with Φ.

We say a flow Φ is *projectively Anosov* (*or* **P**A) when it admits a **P**A splitting (E^u, E^s). If both E^u and E^s are (C^r-)smooth, then Φ is called a (C^r-)regular **P**A flow. In such a case, E^u and E^s generate C^r foliations which are called *the unstable and the stable foliations*, respectively.

2.2. A dichotomy on dynamics. Now, we give a sketch of the proof of the main theorem. Fix a regular **P**A flow on a three dimensional connected and closed manifold M. Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be the stable and unstable foliations respectively. Let $\mathrm{Per}(\Phi)$ denote the set of all periodic point of Φ.

The proof of the main theorem is divided into two parts. First, we show the following dichotomy on dynamics of Φ.

Proposition 2.1. *Either $M = \overline{\mathrm{Per}(\Phi)}$ or Φ is represented by a finite union of $\mathbf{T}^2 \times I$-models.*

Sketch of Proof. The proof is essentially the same as in [2]. However, lack of hyperbolicity of periodic orbits creates additional difficulties in the proof.

By using a standard technique of local return maps, we can apply the argument in the proof of Proposition 3.1 of [15] and obtain the "Denjoy property" for local return maps, which allows us to show the following lemma:

Lemma 2.2. *Let C be a periodic orbit of Φ and $W^s(C)$ the stable set of C. Take a leaf L of $\mathcal{F}^s$ and a connected component U of $(L \cap W^s(C)) \setminus C$. If U is not empty, then, with the leaf topology of L, U is homeomorphic to $S^1 \times \mathbf{R}$ and the boundary of U consists of periodic orbits.*

For a periodic orbit C of Φ, let $\mathcal{F}^s(C)$ and $\mathcal{F}^u(C)$ denote the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ which contain C. By the same argument as in [2], if $\mathcal{F}^s$ has contracting linear holonomy along the periodic point C then $W^s(C) = \mathcal{F}^s(C)$ and it is homeomorphic to $S^1 \times \mathbf{R}$. By applying the level theory [4] and the stability theory of non-compact leaves [5] of Cantwell and Conlon, we also obtain that $W^u(C) = \mathcal{F}^u(C)$ or C is contained in a Φ-invariant torus consisting of periodic orbits.

The "Denjoy property" of local return maps also implies the existence of a local product structure on $\overline{\mathrm{Per}(\Phi)}$. Therefore, we can apply the same argument as in [2], which completes the proof of Proposition 2.1. □

By Proposition 2.1, we have only to show the following.

Proposition 2.3. *If $M = \overline{\mathrm{Per}(\Phi)}$, then Φ is an Anosov flow.*

Sketch of Proof. By a theorem of Arroyo and Rodriguez Hertz [1, Theorem B], it is sufficient to show that all periodic orbits are hyperbolic.

First, we consider the case that Φ has a global cross section M_0. Remark that M_0 is diffeomorphic to $\mathbf{T}^2$. Let $F : M_0 \to M_0$ be the global return map of Φ. We put $E^{uu} = TM_0 \cap E^u$ and $E^{ss} = TM_0 \cap E^s$. Let $W^{ss}(z)$ and $W^{uu}(z)$ denote the stable and unstable sets for a point $z \in \mathbf{T}^2$. Since there exists a local product structure on M_0, $W^{ss}(z)$ and $W^{uu}(z)$ are leaves of the foliation generated by E^{ss} and E^{uu} respectively and there exists a Markov partition for F.

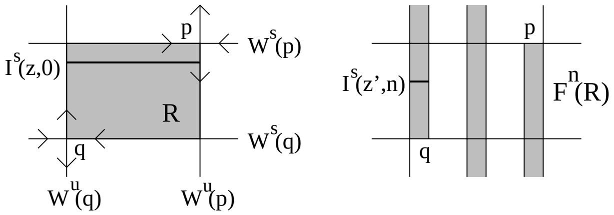

Fig. 1. The rectangle R.

Suppose $p \in M_0$ is a non-hyperbolic periodic point. Without loss of generality, we can assume that p is a fixed point and $\|DF|_{E^{ss}}\| = 1$. By the existence of Markov partition, we can reduce F to a one dimensional dynamical sy stem. A theorem of Mañé [8, 9] implies that the number of non-hyperbolic periodic orbits is finite. Take a hyperbolic periodic point q close to p and a small rectangle $R \subset M_0$ as in Figure 1. For $n \geq 0$ and $z \in F^n(R)$, let $I^s(z,n)$ be the connected component of $W^s(z) \cap F^n(R)$ which contains z. Notice that the existence of a Markov partition implies that if we choose sufficiently small R then $\{I^s(z,n) \mid n \geq 0,\, z \in F^n(R)\}$ is bounded by a small number. By replacing F by its iteration, we can assume that q is a fixed point.

For a C^2 one dimensional map $g : I \to I'$, we define the distortion $\mathrm{Dist}(g)$ of g by

$$\mathrm{Dist}(g) = \sup\{\log|Dg(x)/Dg(x')| \mid x,\, x' \in I\}.$$

Let $h_n : I^s(p,n) \to I^s(q,n)$ be the holonomy map of the foliation generated by E^{uu} in $F^n(R)$. Since the area of $F^n(R)$ is bounded by that of M_0 and $\{I^s(z,n)\}$ is bounded by a small number, a standard method of the theory of C^2 codimension one foliation implies that $\{\mathrm{Dist}(h_n)\}$ is bounded. By the hyperbolicity of fixed point q, we also obtain that $\{\mathrm{Dist}(F^n|_{I^s(q,0)})\}$ is bounded. On the other hand, $\{\mathrm{Dist}(F^n|_{I^s(p,0)})\}$ is unbounded since $\|DF^n|_{E^{ss}(p)}\| = 1$ for any $n \geq 1$ and the length of $F^n(I^s(p,0))$ goes to zero as $n \to \infty$. It contradicts $h_n \circ (F^n|_{I^s(p,0)}) = (F^n|_{I^s(q,0)}) \circ h_0$. Therefore, all periodic points are hyperbolic if Φ has a global cross section.

When Φ has no global cross section, we show that there exist a time change Φ' of Φ and a splitting $TM = T\Phi' \oplus E^{ss} \oplus E^{uu}$ which is "invariant" under Φ' in some sense. For such a flow Φ', we can apply an argument similar to the above, which show that Φ' and Φ are Anosov flows. □

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References

[1] Arroyo, A., and Rodriguez Hertz, F.: Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows. Ann. Inst. H. Poincaré Anal. Non Linéaire, **20**, 805–841 (2003).

[2] Asaoka, M.: Classification of regular and non-degenerate projectively Anosov diffeomorphisms on three dimensional manifolds. (Preprint). http://www.math.kyoto-u.ac.jp/~asaoka/papers/regular.pdf

[3] Asaoka, M.: Regular projectively Anosov flows on 3-manifolds. (In preparation).

[4] Cantwell, J., and Conlon, L.: The theory of levels. Index theory of elliptic operators, foliations, and operator algebras (New Orleans, LA/Indianapolis, IN, 1986), Contemp. Math., 70, Amer. Math. Soc., Providence, RI, pp. 1–10 (1988).

[5] Cantwell, J., and Conlon, L.: Endsets of exceptional leaves; a theorem of G.Duminy. Foliations: geometry and dynamics (Warsaw 2000), World Sci. Publishing, River Edge, NJ, pp. 225–261 (2002).

[6] Eliashberg, Y., and Thurston, W.: Confoliations. University Lecture Series, 13. Amer. Math. Soc., Providence, RI (1998).

[7] Ghys, E.: Rigidité différentiable des groupes fuchiens. Inst. Hautes Études Sci. Publ. Math., **78**, 163–185 (1993).

[8] Mañé, R.: Hyperbolicity, sinks and measure in one-dimensional dynamics. Comm. Math. Phys., **100**(4), 495–524 (1985).

[9] Mañé, R.: Erratum: "Hyperbolicity, sinks and measure in one-dimensional dynamics." Comm. Math. Phys., **112**(4), 721–724 (1987).

[10] Mitsumatsu, Y.: Anosov flows and non-stein symplectic manifolds. Ann. Inst. Fourier (Grenoble), **45**(5), 1407–1421 (1995).

[11] Mitsumatsu, Y.: Foliations and contact structures on 3-manifolds. Foliations: geometry and dynamics (Warsaw 2000), World Sci. Publishing, River Edge, NJ, pp. 75–125 (2002).

[12] Noda, T.: Projectively Anosov flows with differentiable (un)stable foliations. Ann. Inst. Fourier (Grenoble), **50**(5), 1617–1647 (2000).

[13] Noda, T.: Regular projectively Anosov flows with compact leaves. Ann. Inst. Fourier (Grenoble), **54**(2), 353–363 (2004).

[14] Noda, T., and Tsuboi, T.: Regular projectively Anosov flows without compact leaves. Foliations: geometry and dynamics, (Warsaw 2000), World Sci. Publishing, River Edge, NJ, pp. 403–419 (2002).
[15] Pujals, E., and Sambarino, M.: Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. Ann. of Math. (2), **151**(3), 961–1023 (2000).
[16] Tsuboi, T.: Regular projectively Anosov flows on the Seifert fibered 3-manifolds. J. Math. Soc. Japan, **56**(4), 1233–1253 (2004).