Missing terms in Hardy-Sobolev inequalities

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Abstract: In this article we shall investigate Hardy-Sobolev inequalities and improve them by adding a term with a singular weight of the type $(\log(1/|x|))^{-2}$. We show that this weight function is optimal in the sense that the inequality fails for any other weight more singular than this one. As an application, we use our improved inequality to study the weighted eigenvalue problem stated in §5.

Key words: Hardy-Sobolev inequality; eigenvalue.

1. Introduction. In this paper, we shall study the Hardy-Sobolev inequalities of the following type:

(1.1)
$$\int_\Omega |\Delta u|^p dx \geq \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx$$

for any $u \in W_0^{2,p}(\Omega)$, where Ω a bounded domain in $\mathbf{R}^n$ with $0 \in \Omega$, $n \geq 3$, and $1 < p < (n/2)$. Here the best constant $\Lambda_{n,p} = ((n-2p)/p)^p\,((np-n)/p)^p$ is given by the infimum of

$$I(u) = \frac{\displaystyle\int_\Omega |\Delta u|^p dx}{\displaystyle\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx}.$$

Moreover there exists no extremal function in $W_0^{2,p}(\Omega)$ which attains the infimum of these problem. Roughly speaking, the candidates of the extremals are singular at the origin which are not in the class $W_0^{2,p}(\Omega)$. Hence it is natural to consider that there exist "missing terms" in the right hand side of (1.1). In view of this, we shall investigate the Hardy-Sobolev inequalities (1.1) and improve them by finding out missing terms.

For the case of gradient, such improved Hardy inequalities are known. For example, Adimurthi, Chaudhuri and Ramaswamy [1] have proved that there exists a constant $C > 0$, depending on $n \geq 2, 1 < p < n$ and $R > \sup_\Omega(|x|e^{(2/p)})$ such that for $u \in W_0^{1,p}(\Omega)$

$$(1.2)\quad \int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} dx + C\int_\Omega \frac{|u(x)|^p}{|x|^p}\left(\log\frac{R}{|x|}\right)^{-\gamma} dx.$$

Our aim in this article is to achieve an optimal improvement of the inequality (1.1) by adding a second term involving the singular weight $(\log(1/|x|))^{-2}$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one.

As an application, we use our improved inequality to determine exactly when the first eigenvalue of the weighted eigenvalue problem for the operator

$$(1.3)\qquad L_\mu u = \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \frac{\mu}{|x|^{2p}}|u|^{p-2}u$$

will tend to 0 as μ increases to $\Lambda_{n,p}$.

This paper is organized in the following way. In §2 we shall describe our main results on Hardy-Sobolev inequalities. In §3 we shall prepare lemmas which are needed in the proof of the theorem stated in §2. In §4 we shall prove the main results (Theorem 2.1 and Corollary 2.1). In §5 we shall apply our results to study the weighted eigenvalue problem.

This paper is a resume of the paper entitled *Missing terms in Hardy-Sobolev inequalities and its applications* submitted to *Far East Journal of Mathematical Sciences* [2]. Hence the proof of the theorems were given in a shorter way.

2. Main results.

Theorem 2.1 (main). *Let* $n \geq 3$, $0 \in \Omega$ *and* Ω *is a bounded domain in* $\mathbf{R}^n$.

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1. Noncritical case ($1 < p < (n/2)$).
   Assume $\gamma \geq 2$, then there exists $K = K(n) > 0$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then

$$\int_\Omega |\Delta u|^p dx \geq \left(\frac{n-2p}{p}\right)^p \left(\frac{np-n}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx + C \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \tag{2.1}$$

   for any $u \in W_0^{2,p}(\Omega)$.
2. Critical case ($p = (n/2)$).
   Assume $\gamma \geq (n/2)$, then there exists $K^* = K^*(n) > 0$ and $C^* = C^*(n) > 0$ such that if $R > K^* \sup_\Omega |x|$ then

$$\int_\Omega |\Delta u|^{\frac{n}{2}} dx \geq \left(\frac{n-2}{\sqrt{n}}\right)^n \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx + C^* \int_\Omega \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\gamma-1} dx \tag{2.2}$$

   for any $u \in W_0^{2,(n/2)}(\Omega)$.

Remark 2.1. In (2.1) $\gamma \geq 2$ is sharp. In (2.2) $\gamma \geq (n/2)$ is also sharp, and $((n-2)/(\sqrt{n}))^n$ is best constant.

Remark 2.2. The function $\gamma \to (\log(R/r))^{-\gamma}$ is monotonically decreasing on $[2, \infty)$ ($[(n/2), \infty)$) provided that $R > \sup_\Omega |x|$. Hence it suffices to assume $\gamma = 2$ ($\gamma = (n/2)$) in noncritical case (critical case).

Remark 2.3. In the proof of the noncritical case, we will use decreasing rearrangement argument, hence the function $g(r) = r^{-2p} (\log(R/r))^{-2}$ should be monotone decreasing and $R \geq re^{(1/p)}$. Then $K = e^{(1/p)}$.

Remark 2.4. In the proof of the critical case, we will also use decreasing rearrangement argument, hence the function $g^*(r) = r^{-n} (\log(R/r))^{-(n/2)-1}$ should also be monotone decreasing and $R \geq re^{(1/2)+(1/n)}$. Moreover we need the condition to absorb the error terms in the right hand side of (2.2) with $C^* > 0$, hence $K^* \geq e^{(1/2)+(1/n)}$.

Remark 2.5. C and C^* may depend on R and γ in a weak sense. Since $g(r)$ and $g^*(r)$ tends to zero as $\gamma \to \infty$ or $R \to \infty$, therefore we can take C and C^* to be bigger.

Corollary 2.1. *Let $1 < p < (n/2)$, and let*

$$F_p = \left\{ f : \Omega \to \mathbf{R}^+ | f \in L^\infty_{loc}(\Omega \setminus \{0\}) \text{ with } \limsup_{|x| \to 0} |x|^{2p} f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}.$$

If $f \in F_p$, then there exists $\lambda(f) > 0$ such that for $u \in W_0^{2,p}(\Omega)$

$$\int_\Omega |\Delta u|^p dx \geq \Lambda_{n,p} \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx + \lambda(f) \int_\Omega |u(x)|^p f(x) dx. \tag{2.3}$$

If $f \notin F_p$ and if $|x|^{2p} f(x) (\log(1/|x|))^2$ tends to ∞ as $|x| \to 0$, then no inequality of type (2.3) can hold.

3. Preliminary Lemmas.

In this section we shall prepare fundamental lemmas which are needed to prove our main results.

Lemma 3.1. *For any $R > 1$, $q \leq 0$, $\nu \in (0,1)$ satisfying $2\nu - 1 + q = 0$.*

$$\int_0^1 |h'(r)|^2 \left(\log \frac{R}{r}\right)^q r dr \geq \nu^2 \int_0^1 |h(r)|^2 \left(\log \frac{R}{r}\right)^{q-2} \frac{dr}{r} \tag{3.1}$$

holds for any $h \in C([0,1]) \cap C^1(0,1)$, with $h(0) = h(1) = 0$.

Lemma 3.2. *Assume $f \in C^2(B_1)$ and $u \in C_0^2(B_1)$ are radial satisfying $f(r) > 0$, $\Delta f(r) \leq 0$, $u(r) > 0$, and $-\Delta u > 0$ where $r = |x|$. Set $u(r) = f(r)v(r)$, then*

$$\begin{aligned} \int_{B_1} |\Delta u|^{\frac{n}{2}} dx \geq{}& \omega_n \int_0^1 r^{n-1} |v(r) \Delta f(r)|^{\frac{n}{2}} dr \\ &+ \frac{n(n-2)}{4} \omega_n \int_0^1 \left(v'(r)\right)^2 v^{\frac{n-4}{2}}(r) r^{n-1} |\Delta f(r)|^{\frac{n-2}{2}} f(r) dr \\ &+ \omega_n \int_0^1 v^{\frac{n}{2}}(r) \partial_r \Big(r^{n-1} \big(|\Delta f(r)|^{\frac{n-2}{2}} f'(r) - \partial_r |\Delta f(r)|^{\frac{n-2}{2}} f(r) \big) \Big) dr. \end{aligned} \tag{3.2}$$

We can now prove the critical case of Theorem 2.1 using $\Omega = B_1$ and u is radial.

Lemma 3.3. *Assume positive radially nonincreasing function $u \in C_0^2(B_1)$, then there exists $K^* = K^*(n) > 0$ and $C^* = C^*(n) > 0$ such that if $R > K^*$*

$$(3.3)\quad \int_{B_1} |\Delta u|^{\frac{n}{2}} dx \geq \left(\frac{n-2}{\sqrt{n}}\right)^n \int_{B_1} \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\frac{n}{2}} dx + C^* \int_{B_1} \frac{|u(x)|^{\frac{n}{2}}}{|x|^n} \left(\log \frac{R}{|x|}\right)^{-\frac{n}{2}-1} dx.$$

A sketch of the proof of Lemma 3.3. For $u(r) = f(r)v(r)$, we set $f(r) = (\log(R/r))^a$, $0 < a < 1$. Then we use Lemma 3.1 and Lemma 3.2 to prove inequality (3.3).

4. Proof of main results. We recall the rearrangement of domains and functions. For a domain Ω we define a ball Ω^* such that $|\Omega^*| = |\Omega|$ with center at the origin. We denote by $|u|^*$ the symmetric decreasing rearrangement of function u. It is well known that the symmetric rearrangement does not change the L^p-norm and increases the integral $\int_\Omega (|u|^p)/(|x|^{2p})dx$.

The following is due to G. Talenti. As for the proof refer to [4] and [6]

Lemma 4.1 (Talenti). *Let Ω be a domain on $\mathbf{R}^n$, $n \geq 3$ and $f \in C_0^\infty(\Omega)$. If u is the weak solution of Dirichlet problem $-\Delta u = f$ in Ω, $u|_{\partial\Omega} = 0$; v is the weak solution of Dirichlet problem $-\Delta v = |f|^*$ in Ω^*, $v|_{\partial\Omega^*} = 0$; then $v \geq |u|^*$ pointwise.*

From this lemma we see that $u^* \leq v$ in Ω^* and

$$\int_\Omega |\Delta u|^p dx = \int_\Omega |f|^2 dx = \int_{\Omega^*} |f|^{*2} dx \int_{\Omega^*} |\Delta v|^p dx.$$

Further we see that

$$\int_\Omega \frac{|u|^p}{|x|^{2p}} dx \leq \int_{\Omega^*} \frac{|u^*|^p}{|x|^{2p}} dx \leq \int_{\Omega^*} \frac{|v|^p}{|x|^{2p}} dx.$$

Hence we have

$$\frac{\int_\Omega |\Delta u|^p dx}{\int_\Omega \frac{|u|^p}{|x|^{2p}} dx} \geq \frac{\int_{\Omega^*} |\Delta v|^p dx}{\int_{\Omega^*} \frac{|v|^p}{|x|^{2p}} dx}.$$

From this we can assume that u is radial and Ω is a ball in the proof of the main results.

We are now ready to give the proof of Theorem 2.1. We organize the proof in the following way: First for noncritical case $(1 < p < (n/2))$, we prove inequality (2.1), then we show the sharpness of $\Lambda_{n,p}$, and then we show the optimality of γ. Secondly for critical case $(p = (n/2))$, we prove inequality (2.2), then we show the sharpness of $((n-2)/(\sqrt{n}))^n$, and then we show the optimality of γ.

Proof of Theorem 2.1.

Case $1 < p < (n/2)$: Let $1 < p < (n/2)$ and $\gamma \geq 2$. It suffices to prove (2.1) for smooth positive radially nonincreasing function u defined on a ball B_1 and from Remark 2.2 we can further assume $\gamma = 2$. For $u \in C_0^\infty(B_1)$, $u > 0$, radially nonincreasing, we define

$$(4.1)\qquad v(r) = u(r) r^{\frac{n}{p}-2}, \quad r = |x|.$$

From Lemma 4.1 we may assume $-\Delta u > 0$.

$$\Delta(u(r)) = \Delta\left(v(r) r^{2-\frac{n}{p}}\right) = r^{2-\frac{n}{p}} \left(\delta_\beta(v(r)) - \alpha \frac{v(r)}{r^2}\right)$$

where

$$\delta_\beta(v(r)) = \Delta_\beta(v(r)) = v''(r) + \frac{\beta-1}{r} v'(r),$$

$$\beta = n + 4 - \frac{2n}{p} \quad \text{and} \quad \alpha = \frac{(n-2p)(np-n)}{p^2}.$$

Then by using inequality $(1+x)^p \geq 1 + px$ $(x > -1)$ and Lemma 3.1 $(\nu = (1/2), q = 0)$ we get inequality (2.1) where $C = (\Lambda_{n,p})/(\alpha)(p-1)/p$.

We construct a family of functions in $W_0^{2,p}(B_1)$ which we will use to show the sharpness of $\Lambda_{n,p}$ and optimality of γ. For $\epsilon > 0$ sufficiently small, let us define

$$(4.2)\qquad u_\epsilon = \begin{cases} 0, & 0 < r < \epsilon^2 \\ r^{1-\frac{n}{p}} \left(\log \frac{1}{\epsilon}\right)^{-1} \log \frac{r}{\epsilon^2}, & \epsilon^2 < r < \epsilon \\ r^{1-\frac{n}{p}} \left(\log \frac{1}{\epsilon}\right)^{-1} \log \frac{1}{r}, & \epsilon < r < 1. \end{cases}$$

Let $w_\epsilon = \int_r^1 u_\epsilon(\rho) d\rho$. Then we get

$$(4.3)\quad \int_{B_1} |\Delta w_\epsilon|^p dx = \frac{2}{p+1} \left(\frac{n(p-1)}{p}\right)^p \omega_n \log \frac{1}{\epsilon} + O\left(\left(\log \frac{1}{\epsilon}\right)^{-1}\right).$$

$$(4.4)\quad \int_{B_1} \frac{|w_\epsilon|^p}{|x|^{2p}} dx \geq \frac{2}{p+1} \left(\frac{p}{n-2p}\right)^p \omega_n \log \frac{1}{\epsilon} + O\left(\left(\log \frac{1}{\epsilon}\right)^{-1}\right).$$

Also we get

$$(4.5)\quad \int_{B_1} \frac{|w_\epsilon|^p}{|x|^{2p}} \left(\log \frac{R}{|x|}\right)^{-\gamma} \geq O\left(\left(\log \frac{1}{\epsilon}\right)^{1-\gamma}\right).$$

Sharpness of $\Lambda_{n,p}$. The sharpness of $\Lambda_{n,p}$ will follow if we can show that

$$\inf_{u\in W_0^{2,p}(B_1)\setminus\{0\}} I(u) := \frac{\int_{B_1}|\Delta u|^p dx}{\int_{B_1}\frac{|u|^p}{|x|^{2p}}dx} = \Lambda_{n,p}. \tag{4.6}$$

Using the family of functions $w_\epsilon \in W_0^{2,p}(B_1)$, and from (4.3) and (4.4) we get

$$\lim_{\epsilon\to 0} I(w_\epsilon) \le \Lambda_{n,p}.$$

Also by Hardy inequality we get $\lim_{\epsilon\to 0} I(w_\epsilon) \ge \Lambda_{n,p}$, hence $\lim_{\epsilon\to 0} I(w_\epsilon) = \Lambda_{n,p}$. Thus sharpness follow.

Optimality of γ. Suppose $0 \le \gamma < 2$. Since $\Lambda_{n,p}$ is the best constant for inequality (1.1) inequality (2.1) follows for the case $\gamma = 0$. So we assume $0 < \gamma < 2$. Optimality will follow if we can prove that

$$\inf_{u\in W_0^{2,p}(B_1)\setminus\{0\}} I_\gamma(u) = \frac{\int_{B_1}|\Delta u|^p dx - \Lambda_{n,p}\int_{B_1}\frac{|u|^p}{|x|^{2p}}dx}{\int_{B_1}\frac{|u|^p}{|x|^{2p}}\left(\log\frac{R}{|x|}\right)^{-\gamma}dx} = 0. \tag{4.7}$$

Since $0 < \gamma < 2$, from (4.3), (4.4) and (4.5), $I_\gamma(w_\epsilon) \to 0$ as $\epsilon \to 0$ and hence optimality follow.

Case $p = (n/2)$**:** Let us assume $p = (n/2)$ and $\gamma \ge (n/2)$. From Remark 2.2 we can further assume $\gamma = (n/2)$. In this case inequality (2.2) follows immediately from Lemma 3.3 together with the rearrangement of function and domain arguments.

Sharpness. To show the sharpness of $((n-2)/\sqrt{n}))^n$, we consider the test function

$$z_\epsilon = \left(\log\frac{R}{r+\epsilon}\right)^{\frac{n-2}{n}} - \left(\log\frac{R}{1+\epsilon}\right)^{\frac{n-2}{n}}.$$

Then it is easy to verify by similar calculation as in the previous case $(1 < p < (n/2))$ that

$$\lim_{\epsilon\to 0}\frac{\int_{B_1}|\Delta z_\epsilon|^{\frac{n}{2}}dx}{\int_{B_1}\frac{|z_\epsilon|^{\frac{n}{2}}}{|x|^n}\left(\log\frac{R}{|x|}\right)^{-\gamma}dx} = \left(\frac{n-2}{\sqrt{n}}\right)^n.$$

Hence sharpness follow.

Optimality. To show the optimality, we use the same test function u_ϵ define in (4.2) with $p = (n/2)$. Then for $w_\epsilon = \int_r^1 u_\epsilon(\rho)d\rho$, it is easy to verify by similar calculation as in the previous case $(1 < p < (n/2))$ that

$$\int_{B_1}|\Delta w_\epsilon|^{\frac{n}{2}}dx = \frac{4}{n+2}(n-2)^{\frac{n}{2}}\omega_n\log\frac{1}{\epsilon} + O\left(\left(\log\frac{1}{\epsilon}\right)^{-1}\right). \tag{4.8}$$

and

$$\int_{B_1}\frac{|w_\epsilon|^{\frac{n}{2}}}{|x|^n}\left(\log\frac{R}{|x|}\right)^{-\gamma}dx \ge O\left(\left(\log\frac{1}{\epsilon}\right)^{\frac{n}{2}+1-\gamma}\right). \tag{4.9}$$

Suppose $0 < \gamma < (n/2)$. Optimality will follow if we can prove that

$$\inf_{u\in W_0^{2,\frac{n}{2}}(B_1)\setminus\{0\}} I_\gamma(u) = \frac{\int_{B_1}|\Delta u|^{\frac{n}{2}}dx}{\int_{B_1}\frac{|u|^{\frac{n}{2}}}{|x|^n}\left(\log\frac{R}{|x|}\right)^{-\gamma}dx} = 0.$$

Since $0 < \gamma < (n/2)$, from (4.8) and (4.9), $I_\gamma(w_\epsilon) \to 0$ as $\epsilon \to 0$ and hence optimality follow. $\square$

Proof of Corollary 2.1. If $f \in F_p$, then

$$\lim_{\epsilon\to 0}\sup_{x\in B_\epsilon} f(x)|x|^{2p}\left(\log\frac{1}{|x|}\right)^2 < \infty$$

and hence for sufficiently small ϵ, in B_ϵ

$$f(x) < C|x|^{-2p}\left(\log\frac{1}{|x|}\right)^{-2}.$$

Outside B_ϵ, both are bounded functions and hence C can be chosen so that this inequality holds in Ω. Then (2.3) will follow from (2.1).

If $f \notin F_p$ and if $|x|^{2p}f(x)\,(\log(1/|x|))^2$ tends to ∞ as $|x| \to 0$, then we can write $f(x) = (h(x))/\left(|x|^{2p}(\log(1/|x|))^2\right)$, where $h(x)$ tends to infinity as x tends to 0. Then from the calculation of Theorem (2.1), for $\epsilon > 0$ sufficiently small we get

$$\int_{B_1}\frac{|w_\epsilon|^p h(x)}{|x|^{2p}\left(\log\frac{1}{|x|}\right)^2}dx \ge O\left(\left(\log\frac{1}{\epsilon}\right)^{-1}\right)m_\epsilon \tag{4.10}$$

where

$$m_\epsilon = \min\left\{\inf_{B_{\epsilon^2}} h(x),\ \inf_{B_\epsilon\setminus B_{\epsilon^2}} h(x),\ \inf_{B_1\setminus B_\epsilon} h(x)\right\}.$$

Since m_ϵ tends to ∞ as $\epsilon \to 0$, we conclude that $I_f(w_\epsilon) \to 0$ as $\epsilon \to 0$ and inequality (2.3) cannot hold for such $f \notin F_p$. $\square$

5. Application. Consider the weighted eigenvalue problem with a singular weight

(5.1)
$$\Delta\left(|\Delta u|^{p-2}\Delta u\right) - \frac{\mu}{|x|^{2p}}|u|^{p-2}u = \lambda |u|^{p-2}uf \quad \text{in } \Omega$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega.$$

Here $f \in \mathcal{F}_p$

$$\mathcal{F}_p = \Big\{ f : \Omega \to \mathbf{R}^+ \mid \lim_{|x|\to 0} |x|^{2p} f(x) = 0 \quad \text{with } f \in L^\infty_{\mathrm{loc}}\left(\bar{\Omega}\setminus\{0\}\right)\Big\},$$

$1 < p < (n/2)$, $0 \le \mu < \Lambda_{n,p}$ and $\lambda \in \mathbf{R}$. We look for a weak solution $u \in W = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of this problem and study the asymptotic behaviour of the first eigenvalues for different singular weights as μ increases to $\Lambda_{n,p}$, after which the operator L_μ is no more bounded from below. Here we define weak solution in the following way.

Definition 5.1. $u \in W$ is said to be a weak solution of (5.1) iff for any $\phi \in C^2(\bar{\Omega})$ with $\phi = 0$ on $\partial\Omega$

$$\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta\phi - \frac{\mu}{|x|^{2p}}|u|^{p-2}u\phi \right) dx = \lambda \int_\Omega |u|^{p-2}uf\phi dx.$$

Lemma 5.1. *For $\mu^* < \Lambda_{n,p}$, and $u \in W$ $^\exists v \in W$ such that $v > 0$ and satisfies*

$$\frac{\int_\Omega |\Delta u|^p dx - \mu^* \int_\Omega \frac{|u|^p}{|x|^{2p}} dx}{\int_\Omega |u|^p f dx} \ge \frac{\int_\Omega |\Delta v|^p dx - \mu^* \int_\Omega \frac{|v|^p}{|x|^{2p}} dx}{\int_\Omega |v|^p f dx}.$$

Remark 5.1. Since λ is first eigenvalue and u is the corresponding eigenfunction, by using Lemma 5.1, we can assume $u > 0$ in Ω. Then by the elliptic regularity theory, u is smooth near the boundary. From the definition of weak solution one can derive the boundary condition of (5.1) by using integration by parts.

Theorem 5.1. *The above problem admits a positive weak solution $u \in W$ for all $1 < p < (n/2)$, $0 \le \mu < \Lambda_{n,p}$, corresponding to the first eigenvalue $\lambda = \lambda^1_\mu(f) > 0$. Moreover, as μ increases to $\Lambda_{n,p}, \lambda^1_\mu(f) \to \lambda(f) \ge 0$ for all $f \in \mathcal{F}_p$ and the limit $\lambda(f) > 0$ if $f \in F_p$. If $f \notin F_p$ and if $|x|^{2p}f(x)\left(\log(1/|x|)\right)^2$ tends to ∞ as $|x| \to 0$, then the limit $\lambda(f) = 0$.*

In order to prove the theorem we need the following results; the first one is a standard result from measure theory (see [5], Chapter 1, Section 4) and the second one is due to Boccardo and Murat [3].

Lemma 5.2. *Let $(g_m)_{m\in\mathbf{N}} \subset L^p(\Omega), 1 \le p < \infty$, be such that, as $m \to \infty$,* (i) *$g_m \rightharpoonup g$ weakly in $L^p(\Omega)$ and* (ii) *$g_m(x) \to g(x)$ a.e. in Ω. Then*

$$\lim_{m\to\infty}\left(||g_m||_p^p - ||g_m - g||_p^p\right) = ||g||_p^p.$$

Lemma 5.3 (see Remark 2.7 in [2])**.**

$$\left(|\Delta u_m|^{p-2}\Delta u_m - |\Delta u|^{p-2}\Delta u\right)\Delta(u_m - u) \ge \begin{cases} s|\Delta(u_m-u)|^p & \text{if } p \ge 2 \\ \dfrac{s|\Delta(u_m-u)|^2}{\left(|\Delta u_m| + |\Delta u|\right)^{2-p}} & \text{if } p \le 2 \end{cases}$$

for some $s > 0$.

Remark 5.2. In the proof of Theorem 5.1 u will be characterize as a solution of variational problem defined in (5.2) and (5.1) becomes Euler-Lagrange equation of this variational problem.

Sketch of the proof of Theorem 5.1.
- We define

$$J_\mu(u) = \int_\Omega \left(|\Delta u|^p - \mu\frac{|u|^p}{|x|^{2p}}\right) dx. \tag{5.2}$$

- We minimize J_μ over

$$M = \left\{ u \in W \Big| \int_\Omega |u(x)|^p f(x) dx = 1 \right\}$$

and let $\lambda^1_\mu > 0$ be the infimum.
- We choose minimizing sequence $(u_m)_{m\in\mathbf{N}} \subset M$ such that $J_\mu(u_m) \to \lambda^1_\mu$.
- For a subsequence u_{m_k} of u_m

$$\begin{cases} J_\mu(u_{m_k}) \to J_\mu(u) = \lambda^1_\mu = \lambda \\ J'_\mu(u_{m_k}) \to 0 \end{cases}$$

where $u \in W \cap M$
- u satisfies Euler-Lagrange equation in a distribution sense, and since $u \in W$, it is a weak solution of (5.1).
- The remaining part of the proof follows from the corollary of the main theorem. □

References

[1] Adimurthi, Chaudhuri, N., and Ramaswamy, M.: An improved Hardy-Sobolev inequality and its application. Proc. Amer. Math. Soc., **130**(2), 489–505 (2002).

[2] Detalla, A., Horiuchi, T., and Ando, H.: Missing terms in Hardy-Sobolev inequalities and its application. Far East J. Math. Sci., **14**(3), 333-359 (2004).

[3] Boccardo, L., and Murat, F.: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal. **19**(6), 581–597 (1992).

[4] Horiuchi, T.: Missing terms in Generalized Hardy's inequalities and its applications. J. Math. Kyoto Univ., **43**(2), 235–260 (2003).

[5] Struwe, M.: Variational Methods. Applied to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (1996).

[6] Talenti, G.: Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. Ann. Mat. Pura Appl. (4), **120**, 160–184 (1979).