Trace identities of twisted Hecke operators on the spaces of cusp forms of half-integral weight

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(Communicated by Shigefumi MORI, M. J. A., Sept. 13, 2004)

Abstract: Let R_ψ be a twisting operator for a quadratic primitive character ψ and $\tilde{T}(n^2)$ the n^2-th Hecke operator of half-integral weight. When ψ has an odd conductor, we already found trace identities between twisted Hecke operators $R_\psi\tilde{T}(n^2)$ of half-integral weight and certain Hecke operators of integral weight for almost all cases (cf. [U1–3]). In this paper, the restriction is removed and we give similar trace identities for every quadratic primitive character ψ, including the case that ψ has an even conductor.

Key words: Trace identity; twisting operator; half-integral weight; Hecke operator; cusp form.

1. Introduction. Let k, A, and N be positive integers with $4 \mid N$. We denote the space of cusp forms of weight $2k$, level A and the trivial character by $S(2k, A)$. Let χ be an even quadratic character defined modulo N. We denote the space of cusp forms of weight $k+1/2$, level N, and character χ by $S(k+1/2, N, \chi)$.

In [Sh], Shimura had found "Shimura Correspondence". That is an important correspondence from Hecke eigenforms in $S(k+1/2, N, \chi)$ to those in $S(2k, N/2)$.

From the existence of Shimura Correspondence, we can expect that there exist certain identities between traces of Hecke operators of weight $k+1/2$ and those of weight $2k$.

After pioneering works of Niwa [N] and Kohnen [K], we had generalized their results and had found such identities between traces of Hecke operators for almost all levels N (cf. [U1], [U3]). Furthermore, we generalized these results for the *twisted Hecke operators* ([U2]).

We explain more precisely. Let ψ, R_ψ, and $\tilde{T}(n^2)$ be the same as the abstract. In the papers [U1], [U2], and [U4], we calculated the traces of twisted Hecke operators $R_\psi\tilde{T}(n^2)$ both on $S(k+1/2, N, \chi)$ and on Kohnen's plus space $S(k+1/2, N, \chi)_K$. Moreover, when the conductor of ψ is *odd*, we found that the above traces are linear combinations of the traces of certain Hecke operators on the spaces $S(2k, N')$ (N' runs over positive divisors of $N/2$) for almost all cases. However we missed the cases such that $\mathrm{ord}_2(N)$ (the 2-adic additive valuation of N) is equal to 6 and the conductor of χ is divisible by 8.

The purpose of this paper is to remove the above restriction. Namely, we report trace identities for *all* quadratic primitive characters ψ, including both the above missing cases of odd conductors and the cases of even conductors. Details will appear in [U5].

2. Notation. The notation in this paper is the same as in the previous paper [U1]. Hence see [U1] and [U2] for the details of notation. Here, we explain several notations for convenience.

Let k, N, χ be the same as above. For a prime number p, let $\mathrm{ord}_p(\cdot)$ be the p-adic additive valuation with $\mathrm{ord}_p(p) = 1$ and $|\cdot|_p$ the p-adic absolute value which is normalized with $|p|_p = 1$. For a real number x, $[x]$ means the greatest integer less than or equal to x. Let a be a non-zero integer and b a positive integer. We write $a \mid b^\infty$ if every prime factor of a divides b.

Let ρ be any Dirichlet character. We denote the conductor of ρ by $\mathfrak{f}(\rho)$ and for any prime number p, the p-primary component of ρ by ρ_p. Furthermore we set $\rho_A := \prod_{p\mid A} \rho_p$ for an arbitrary integer A. Here p runs over all prime divisors of A. We denote by $\left(\frac{\cdot}{\cdot}\right)$ the Kronecker symbol. See [M, p. 82] for a definition of this symbol.

2000 Mathematics Subject Classification. Primary 11F37; Secondary 11F25.

Let V be a finite-dimensional vector space over $\boldsymbol{C}$. We denote the trace of a linear operator T on V by $\mathrm{tr}(T;V)$.

Put $\mu := \mathrm{ord}_2(N)$ and $\nu = \nu_p := \mathrm{ord}_p(N)$ for any odd prime number p. Then we decompose $N = 2^\mu M$. Namely, M is the odd part of N.

3. Results. Let ψ be a quadratic primitive character with conductor r. Then we can express the conductor r as follows:

$$\begin{cases} r = 2^u L, \qquad u = 0,\ 2, \text{ and } 3 \\ \text{and } L \text{ is a squarefree positive odd integer.} \end{cases}$$

We consider the following conditions $(*1)$–$(*3)$.

$$(*1) \qquad L^2 \mid M.$$

$$(*2) \qquad L^2 \mid M \quad \text{and} \quad \begin{cases} \mu \geqq 5, & \text{if } \mathfrak{f}(\chi_2) = 8. \\ \mu \geqq 4, & \text{if } \mathfrak{f}(\chi_2) \mid 4. \end{cases}$$

$$(*3) \qquad L^2 \mid M \quad \text{and} \quad \mu \geqq 6.$$

From now on until the end of this paper, we assume the following.

Assumption. *We impose the condition* $(*1)$, $(*2)$, *or* $(*3)$ *according to* $u = 0$, 2, *or* 3 *respectively.*

Now, let R_ψ be the twisting operator of ψ:

$$f = \sum_{n \geqq 1} a(n) q^n \mapsto f \mid R_\psi := \sum_{n \geqq 1} a(n)\psi(n) q^n,$$
$$(q := \exp(2\pi\sqrt{-1}z),\ z \in \boldsymbol{C},\ \mathrm{Im}\, z > 0).$$

Then, from the above conditions $(*1$–$3)$ and the assumption $\psi^2 = \mathbf{1}$, we see that the twisting operator R_ψ fixes the space of cusp forms $S(k+1/2, N, \chi)$ (cf. [Sh, Lemma 3.6]).

In the case of $k = 1$, we need to make a certain modification. It is well-known that the space $S(3/2, N, \chi)$ contains a subspace $U(N;\chi)$ which corresponds to a space of Eisenstein series via Shimura correspondence and which is generated by theta series of special type (cf. [U2, § 0(c)]). Let $V(N;\chi)$ be the orthogonal complement of $U(N;\chi)$ in $S(3/2, N, \chi)$. Then it is also well-known that $V(N;\chi)$ corresponds to a space of cusp forms of weight 2 via Shimura correspondence. Hence we need to consider the subspace $V(N;\chi)$ in place of $S(3/2, N, \chi)$ in the case of $k = 1$. The subspaces $U(N;\chi)$ and $V(N;\chi)$ are fixed by the twisting operator R_ψ (See [U5] for a proof and refer also to [U2, p. 94]). Moreover, the n^2-th Hecke operators $\tilde{T}(n^2)$, $(n, N) = 1$, also fix the subspace $V(N;\chi)$ (cf. [U1, p. 508]).

Thus for any positive integer n with $(n, N) = 1$, we can consider the twisted Hecke operator $R_\psi \tilde{T}(n^2)$ on the spaces $S(k+1/2, N, \chi)$ $(k \geqq 2)$ and $V(N;\chi)$ $(k = 1)$ (cf. [U2, p. 86]).

For the statement of Theorem, we prepare a little more notation.

First we decompose the level N with respect to L as follows:

$$N = 2^\mu L_0 L_2, \quad L_0 > 0, \quad L_2 > 0,$$
$$\mu := \mathrm{ord}_2(N), \quad L_0 \mid L^\infty, \quad (L_2, L) = 1.$$

And we put

$$N_0 := \prod_{p \mid L} p^{2[(\nu_p - 1)/2]+1}.$$

Here p runs over all prime divisors of L.

Next, let A be any positive integer. For any odd prime number p and any integers a, b $(0 \leqq a \leqq \mathrm{ord}_p(A)/2)$, we put

$$\lambda_p(\chi_p, \mathrm{ord}_p(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{-b}{p}\right), & \text{if } 1 \leqq a \leqq \left[(\mathrm{ord}_p(A) - 1)/2\right], \\ \chi_p(-b), & \text{if } \mathrm{ord}_p(A) \text{ is even} \\ & \text{and } a = \mathrm{ord}_p(A)/2 \geqq 1. \end{cases}$$

And for any integers a, b $(0 \leqq a \leqq \mathrm{ord}_2(A)/2)$, we put

$$\lambda_2(\chi_2, \mathrm{ord}_2(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{if } a = 1, \\ \xi(b)\left(1 + \left(\frac{2}{b}\right)\right), & \\ & \text{if } 2 \leqq a \leqq \left[(\mathrm{ord}_2(A) - 1)/2\right], \\ \xi(b)\chi_2(-b), & \text{if } \mathrm{ord}_2(A) \text{ is even} \\ & \text{and } a = \mathrm{ord}_2(A)/2 \geqq 2. \end{cases}$$

Here, $\xi(b) := \left(1 - \left(\frac{-1}{b}\right)\right)/2$.

Then for any integer b and any square integer c, we put

$$\Lambda_\chi(\psi, A; b, c) := \prod_{\substack{p \mid A \\ (p, r) = 1}} \lambda_p(\chi_p, \mathrm{ord}_p(A); b, \mathrm{ord}_p(c)/2).$$

Here p runs over all prime divisors of A prime to r.

Furthermore, let B be a positive integer such that $B \mid r^\infty$ and $(A/B, B) = 1$. For all positive integers n such that $(n, N) = 1$, we define

$$\Theta_\psi[2k, n; A, B, \chi] = \Theta_\psi[A, B, \chi]$$
$$:= \sum_{\substack{0<N_1|A \\ N_1=\square,\ (N_1,r)=1}} \Lambda_\chi(\psi, A; rn, N_1) \times \operatorname{tr}(W(BN_1)T(n); S(2k, N_1N_2)),$$

where N_1 runs over all square divisors of A which are prime to r and $N_2 := A\prod_{p|N_1}|A|_p$.

Remark. All the spaces which occur in the definition of $\Theta_\psi[A, B, \chi]$ are contained in the space $S(2k, A)$.

Finally, let χ_r be the r-primary component of χ and $\chi'_r := \prod_{p|N,(p,r)=1}\chi_p$, where p runs over all prime divisors of N which are prime to r. Then we put

$$c(k, n; \psi, \chi) = c(\psi, \chi) := \psi(-1)^k\chi_r(n)\chi'_r(-r).$$

Under these notations, we can state trace identities of the twisted Hecke operators $R_\psi\tilde{T}(n^2)$.

First we state trace identities for the case of *odd* conductors.

Theorem 1. *Let k, N, and χ be the same as above. Suppose that ψ is a quadratic primitive character defined modulo an odd positive integer r. Hence we assume the condition* $(*1)$.

For all positive integers n such that $(n, N) = 1$, we have the following trace identities.

(1) Suppose that $\mu = 2$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, N, \chi)_K\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(N;\chi)_K\big) & \text{if } k = 1 \end{matrix}\right\} = c(\psi,\chi)\Theta_\psi[N_0L_2, N_0, \chi].$$

(2) Suppose that $2 \leqq \mu \leqq 4$ and furthermore $\mathfrak{f}(\chi_2) = 8$ if $\mu = 4$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, N, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(N;\chi)\big) & \text{if } k = 1 \end{matrix}\right\} = c(\psi,\chi)\Theta_\psi[2^{\mu-1}N_0L_2, N_0, \chi].$$

(3) Suppose that $4 \leqq \mu \leqq 6$ and furthermore $\mathfrak{f}(\chi_2)$ divides 4 if $\mu = 4, 6$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, N, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(N;\chi)\big) & \text{if } k = 1 \end{matrix}\right\} = 2c(\psi,\chi)\Theta_\psi[2^{\mu-2}N_0L_2, N_0, \chi].$$

(4) Suppose that $\mu = 6$ and $\mathfrak{f}(\chi_2) = 8$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^6M, \chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^5M, \chi\left(\tfrac{2}{}\right))\big), & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^6M;\chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^5M;\chi\left(\tfrac{2}{}\right))\big), & \text{if } k = 1. \end{matrix}\right\}$$
$$= 4c(\psi,\chi)\times\left\{\begin{matrix}\Theta_\psi[2^3N_0L_2, N_0, \chi] \\ -\Theta_\psi\left[2^2N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right]\end{matrix}\right\}.$$

(5) Suppose that $\mu = 7$ and $\mathfrak{f}(\chi_2)$ divides 4. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^7M, \chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^6M, \chi\left(\tfrac{2}{}\right))\big), & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^7M;\chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^6M;\chi\left(\tfrac{2}{}\right))\big), & \text{if } k = 1. \end{matrix}\right\}$$
$$= 2c(\psi,\chi)\times\left\{\begin{matrix}\Theta_\psi[2^5N_0L_2, N_0, \chi] \\ -\Theta_\psi\left[2^4N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right]\end{matrix}\right\}.$$

(6) Suppose that $\mu = 7$ and $\mathfrak{f}(\chi_2) = 8$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^7M, \chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^6M, \chi\left(\tfrac{2}{}\right))\big), & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^7M;\chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^6M;\chi\left(\tfrac{2}{}\right))\big), & \text{if } k = 1. \end{matrix}\right\}$$
$$= 2c(\psi,\chi)\times\big\{\Theta_\psi[2^5N_0L_2, N_0, \chi] - \Theta_\psi[2^4N_0L_2, N_0, \chi] - \Theta_\psi\left[2^4N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right] + 2\Theta_\psi[2^3N_0L_2, N_0, \chi] + \Theta_\psi\left[2^3N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right] - 2\Theta_\psi\left[2^2N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right]\big\}.$$

(7) Suppose that $\mu \geqq 8$. We have

$$\left\{\begin{matrix} \operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^\mu M, \chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); S(k+1/2, 2^{\mu-1}M, \chi\left(\tfrac{2}{}\right))\big), & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^\mu M;\chi)\big) - \psi(2)\operatorname{tr}\big(R_\psi\tilde{T}(n^2); V(2^{\mu-1}M;\chi\left(\tfrac{2}{}\right))\big), & \text{if } k = 1. \end{matrix}\right\}$$
$$= 2c(\psi,\chi)\times\left\{\begin{matrix}\Theta_\psi[2^{\mu-2}N_0L_2, N_0, \chi] \\ -\Theta_\psi\left[2^{\mu-3}N_0L_2, N_0, \chi\left(\tfrac{2}{}\right)\right]\end{matrix}\right\}.$$

$\square$

Next, we state trace identities for the case of *even* conductor.

Theorem 2. *Let k, N, and χ be the same as above. Suppose that ψ is a quadratic primitive character defined modulo an even positive integer r. Hence we assume the condition* $(*2)$ *or* $(*3)$ *according to $u = 2$ or 3 respectively.*

For all positive integers n such that $(n, N) = 1$, we have the following trace identities.

$\boxed{\text{Case I. } (u=2)}$ $\left(\Leftrightarrow \psi_2 = \left(\frac{-1}{}\right)\right)$

(I-1) Suppose that $\mu = 4$ and $\mathfrak{f}(\chi_2)$ divides 4. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^4M, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^4M; \chi)\big) & \text{if } k = 1 \end{array}\right\}$$
$$= \chi_2\Big(\left(\frac{-1}{Ln}\right)\Big) c(\psi,\chi)\Theta_\psi[2^2N_0L_2, 2^2N_0, \chi].$$

(I-2) Suppose that $\mu = 5$ and $\mathfrak{f}(\chi_2)$ divides 4. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^5M, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^5M; \chi)\big) & \text{if } k = 1 \end{array}\right\}$$
$$= \chi_2\Big(\left(\frac{-1}{Ln}\right)\Big) c(\psi,\chi) \times \big\{\Theta_\psi[2^3N_0L_2, N_0, \chi]$$
$$- 2\Theta_\psi[2^2N_0L_2, N_0, \chi] + 2\Theta_\psi[2^2N_0L_2, 2^2N_0, \chi]\big\}.$$

(I-3) Suppose that $\mu = 5, 6$ and $\mathfrak{f}(\chi_2) = 8$. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^\mu M, \chi)\big) = 0 & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^\mu M; \chi)\big) = 0 & \text{if } k = 1. \end{array}\right.$$

(I-4) Suppose that $\mu = 7$ and $\mathfrak{f}(\chi_2) = 8$. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^7M, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^7M; \chi)\big) & \text{if } k = 1 \end{array}\right\}$$
$$= \Big(1 - \psi(-1)\left(\frac{-1}{n}\right)\Big) c(\psi,\chi)$$
$$\times \big\{\Theta_\psi[2^6N_0L_2, 2^6N_0, \chi] - \Theta_\psi[2^4N_0L_2, 2^4N_0, \chi]\big\}.$$

(I-5) Suppose that $\underline{\mu \geqq 8}$, or $\underline{\mu = 6, 7 \text{ and } \mathfrak{f}(\chi_2) \text{ divides } 4}$. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^\mu M, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^\mu M; \chi)\big) & \text{if } k = 1 \end{array}\right\}$$
$$= \Big(1 - \psi(-1)\left(\frac{-1}{n}\right)\Big) c(\psi,\chi)$$
$$\times \Theta_\psi[2^{\hat{\mu}-2}N_0L_2, 2^{\hat{\mu}-2}N_0, \chi].$$

Here $\hat{\mu}$ is the greatest *even* integer less than or equal to μ, i.e. $\hat{\mu} = 2\big[\mu/2\big]$.

$\boxed{\text{Case II. } (u=3)}$ $\left(\Leftrightarrow \psi_2 = \left(\frac{\pm 2}{}\right)\right)$

(II-1) Suppose that $\mu = 6, 7$ and $\mathfrak{f}(\chi_2) = 8$. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^\mu M, \chi)\big) = 0 & \text{if } k \geqq 2. \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^\mu M; \chi)\big) = 0 & \text{if } k = 1. \end{array}\right.$$

(II-2) Suppose that $\underline{\mu \geqq 8}$, or $\underline{\mu = 6, 7 \text{ and } \mathfrak{f}(\chi_2) \text{ divides } 4}$. We have

$$\left\{\begin{array}{ll} \operatorname{tr}\big(R_\psi \tilde{T}(n^2); S(k+1/2, 2^\mu M, \chi)\big) & \text{if } k \geqq 2 \\ \operatorname{tr}\big(R_\psi \tilde{T}(n^2); V(2^\mu M; \chi)\big) & \text{if } k = 1 \end{array}\right\}$$
$$= \Big(1 - \psi(-1)\left(\frac{-1}{n}\right)\Big) c(\psi,\chi)$$
$$\times \Theta_\psi[2^{\tilde{\mu}-2}N_0L_2, 2^{\tilde{\mu}-2}N_0, \chi].$$

Here $\tilde{\mu}$ is the greatest *odd* integer less than or equal to μ, i.e. $\tilde{\mu} = 2\big[(\mu-1)/2\big] + 1$. □

4. Concluding remarks. We can expect to establish a theory of newforms by using these trace identities. In fact, we established a theory of newforms in the case of level 2^m. See [U6] for the results.

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