On the hybrid mean value of Gauss sums and generalized Bernoulli numbers

By Huaning Liu and Wenpeng Zhang

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P. R. China

(Communicated by Heisuke Hironaka, M. J. A., June 15, 2004)

Abstract: The main purpose of this paper is using the properties of primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of Gauss sums and generalized Bernoulli numbers, and give a sharper asymptotic formula.

Key words: Gauss sums; generalized Bernoulli number; hybrid mean value.

1. Introduction. Let $q \geq 3$ be an integer, χ denote a Dirichlet character modulo q. For any integer n, the famous Gauss sums $G(n,\chi)$ is defined as following:

$$G(n,\chi) = \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{an}{q}\right),$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$. Especially for $n = 1$, we write

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{a}{q}\right).$$

The various properties and applications of $\tau(\chi)$ appeared in many analytic number theory books (See reference [1]).

Maybe the most important property of $\tau(\chi)$ is that if χ is a primitive character modulo q, then

$$|\tau(\chi)| = \sqrt{q}.$$

If χ is a non-primitive character modulo q, $\tau(\chi)$ also appears many good value distribution properties in some problems of weighted mean value. It might be interesting to study the hybrid mean value of $\tau(\chi)$ and other arithmetical functions.

Let χ be a non-principal Dirichlet character modulo q. The generalized Bernoulli numbers $B_{n,\chi}$ is defined by the following:

$$\sum_{a=1}^{q} \chi(a) \frac{t\mathrm{e}^{at}}{\mathrm{e}^{qt} - 1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n.$$

This sequence of numbers has considerable fascination and importance. The definition and basic properties of generalized Bernoulli numbers can be found in [2].

2000 Mathematics Subject Classification. 11L05, 11B68.

In this paper, we use the properties of primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of Gauss sums and generalized Bernoulli numbers, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *Let $q \geq 3$ be an integer, then for any positive integers n and m we have*

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \tau^m\left(\overline{\chi}\right) B_{n,\chi}^m = \frac{2^{m-1} (n!)^m q^{nm-1} \phi^2(q)}{(-1)^{(n-1)m} (2\pi i)^{nm}} \times \prod_{p \| q} \left(1 - \frac{p^{m-1} - 1}{p^{m-1}(p-1)^2}\right) + O\left(q^{nm+\epsilon}\right),$$

where $\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}}$ denotes the summation over all non-principal characters modulo q, $\prod_{p \| q}$ denotes the product over all prime divisors p of q with $p \mid q$ and $p^2 \nmid q$, $\phi(q)$ is the Euler function, and ϵ is any fixed positive number.

2. Some lemmas. To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *For any integer $q \geq 3$, let χ be a non-primitive character modulo q, and q^* denote the conductor of χ with $\chi \Longleftrightarrow \chi^*$. If $(n,q) > 1$, we have*

$$G(n,\chi) = \begin{cases} \overline{\chi}^*\left(\frac{n}{(n,q)}\right) \chi^*\left(\frac{q}{q^*(n,q)}\right) \mu\left(\frac{q}{q^*(n,q)}\right) \phi(q) \\ \quad \times \phi^{-1}\left(\frac{q}{(n,q)}\right) \tau(\chi^*), \quad q^* = \frac{q_1}{(n,q_1)}; \\ 0, \quad q^* \neq \frac{q_1}{(n,q_1)}, \end{cases}$$

where $\mu(n)$ is the Möbius function, and q_1 is the largest divisor of q that has the same prime factors *with q^*.*

If $(n,q)=1$, then we have

$$G(n,\chi)=\overline{\chi}^*(n)\chi^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\chi^*).$$

Proof. See reference [3]. □

Lemma 2. *Let q and r be integers with $q\geq 3$ and $(r,q)=1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \bmod q}{}^{*}\chi(r)=\sum_{d|(q,r-1)}\mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q)=\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of primitive characters modulo q.

Proof. This is Lemma 3 of [4]. □

Lemma 3. *Let $q=uv$, where $(u,v)=1$, u be a square-full number or $u=1$, v be a square-free number. Then for any positive integers n and m we have*

$$\sum_{d|v}(ud)^m\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}{}^{*}\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\right]^m L^m(n,\overline{\chi})$$

(I)

$$=\frac{q^{m-1}\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right)+O\left(q^{m+\epsilon}\right)$$

and

$$\sum_{d|v}(ud)^m\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}{}^{*}\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\right]^m L^m(n,\overline{\chi})$$

(II)

$$=\frac{q^{m-1}\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right)+O\left(q^{m+\epsilon}\right),$$

where $L(s,\chi)$ is the Dirichlet L-function corresponding to χ.

Proof. We only prove (II), similarly we can deduce (I). Let $\tau_m(r)$ denote the m-th divisor function (i.e. the number of positive integer solutions of the equation $r=r_1r_2\cdots r_m$). Note that $J(u)=\phi^2(u)/u$, if u is a square-full number. Then using the methods of Lemma 5 in [5] and Lemma 2 in this paper we have

$$\sum_{d|v}(ud)^m\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}{}^{*}\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\right]^m L^m(n,\overline{\chi})$$

$$=\sum_{d|v}(ud)^m\sum_{t_1|\frac{v}{d}}\cdots\sum_{t_m|\frac{v}{d}}\sum_{r=1}^{\infty}$$

$$\times\frac{\mu(t_1)\cdots\mu(t_m)\phi(t_1)\cdots\phi(t_m)\tau_m(r)}{t_1^n\cdots t_m^n r^n}$$

$$\times\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}{}^{*}\overline{\chi}(t_1\cdots t_m)\overline{\chi}(r)$$

$$=\frac{1}{2}\sum_{d|v}(ud)^m\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{t_1|\frac{v}{d}}\cdots\sum_{t_m|\frac{v}{d}}\sum_{\substack{r=1\\ t_1\cdots t_m r\equiv 1 \bmod s}}^{\infty}$$

$$\times\frac{\mu(t_1)\cdots\mu(t_m)\phi(t_1)\cdots\phi(t_m)\tau_m(r)}{t_1^n\cdots t_m^n r^n}$$

$$-\frac{1}{2}\sum_{d|v}(ud)^m\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{t_1|\frac{v}{d}}\cdots\sum_{t_m|\frac{v}{d}}\sum_{\substack{r=1\\ t_1\cdots t_m r\equiv -1 \bmod s}}^{\infty}$$

$$\times\frac{\mu(t_1)\cdots\mu(t_m)\phi(t_1)\cdots\phi(t_m)\tau_m(r)}{t_1^n\cdots t_m^n r^n}$$

$$=\frac{1}{2}\sum_{d|v}(ud)^m J(ud)+O\left(q^{m+\epsilon}\right)$$

$$=\frac{u^{m-1}\phi^2(u)}{2}\sum_{d|v}d^m J(d)+O\left(q^{m+\epsilon}\right)$$

$$=\frac{u^{m-1}\phi^2(u)}{2}\sum_{p|v}\left[p^{m-1}(p-1)^2\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right)\right]+O\left(q^{m+\epsilon}\right)$$

$$=\frac{q^{m-1}\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right)+O\left(q^{m+\epsilon}\right).$$

This proves Lemma 3. □

3. Proof of the theorem.

In this section, we complete the proof of the theorem. Let $q\geq 3$ be an integer, and χ be a Dirichlet character modulo q. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_{n,\chi}=q^{n-1}\sum_{a=1}^{q}\chi(a)B_n\left(\frac{a}{q}\right).$$

From Theorem 12.19 of [1] we also have

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\mathrm{e}\,(rx)}{r^n}, \quad \text{if } \ 0 < x \leq 1.$$

Therefore

$$B_{n,\chi} = q^{n-1}\sum_{a=1}^{q}\chi(a)\left[-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\mathrm{e}\left(\frac{ar}{q}\right)}{r^n}\right]$$

$$= -\frac{n!q^{n-1}}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{G(r,\chi)}{r^n}.$$

Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Let q^* denote the conductor of χ with $\chi \Longleftrightarrow \chi^*$, then

$$\tau(\overline{\chi}) = \overline{\chi}^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\overline{\chi}^*) \neq 0$$

if and only if $q^* = ud$, where $d \mid v$. So from Lemma 1 and Lemma 3 we have

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\tau^m(\overline{\chi})\,B^m_{n,\chi}$$

$$= \sum_{d|v}\ \sideset{}{^*}\sum_{\chi \bmod ud}\overline{\chi}^m\left(\frac{v}{d}\right)\mu^m\left(\frac{v}{d}\right)\tau^m(\overline{\chi})$$

$$\times\left[-\frac{n!q^{n-1}}{(2\pi i)^n}\sum_{t|\frac{v}{d}}\frac{\chi\left(\frac{v}{dt}\right)\mu\left(\frac{v}{dt}\right)\phi(q)\tau(\chi)}{t^n\phi\left(\frac{q}{t}\right)}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\overline{\chi}(r)}{r^n}\right]^m$$

$$= \frac{(-1)^m(n!)^m q^{(n-1)m}}{(2\pi i)^{nm}}\sum_{d|v}\ \sideset{}{^*}\sum_{\chi \bmod ud}(ud)^m\chi^m(-1)$$

$$\times\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\overline{\chi}(r)}{r^n}\right]^m$$

$$= \begin{cases} \dfrac{(-1)^m 2^m (n!)^m q^{(n-1)m}}{(2\pi i)^{nm}}\displaystyle\sum_{d|v}(ud)^m \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}} \\ \quad\times\left[\displaystyle\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\right]^m L^m(n,\overline{\chi}), & \text{if } \ 2 \mid n \\ \dfrac{2^m (n!)^m q^{(n-1)m}}{(2\pi i)^{nm}}\displaystyle\sum_{d|v}(ud)^m \sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \\ \quad\times\left[\displaystyle\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\right]^m L^m(n,\overline{\chi}), & \text{if } \ 2 \nmid n \end{cases}$$

$$= \begin{cases} \dfrac{(-1)^m 2^{m-1}(n!)^m q^{nm-1}\phi^2(q)}{(2\pi i)^{nm}}\displaystyle\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right) \\ \quad + O\left(q^{nm+\epsilon}\right), & \text{if } \ 2 \mid n \\ \dfrac{2^{m-1}(n!)^m q^{nm-1}\phi^2(q)}{(2\pi i)^{nm}}\displaystyle\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right) \\ \quad + O\left(q^{nm+\epsilon}\right), & \text{if } \ 2 \nmid n \end{cases}$$

$$= \frac{2^{m-1}(n!)^m q^{nm-1}\phi^2(q)}{(-1)^{(n-1)m}\,(2\pi i)^{nm}}\prod_{p\|q}\left(1-\frac{p^{m-1}-1}{p^{m-1}(p-1)^2}\right)$$

$$+ O\left(q^{nm+\epsilon}\right).$$

This completes the proof of the theorem.

Acknowledgement. This work is supported by the N.S.F. and P.N.S.F. of P. R. China.

References

[1] Apostol, T. M.: Introduction to Analytic Number Theory. Springer-Verlag, New York (1976).

[2] Leopoldt, A. W.: Eine Verallgemeinerung der Bernoullischen Zahlen. Abh. Math. Sem. Univ. Hamburg, **22**, 131–140 (1958).

[3] Pan, C.-D., and Pan, C.-B.: Goldbach's Conjecture. Science Press, Beijing (1981), (in Chinese).

[4] Zhang, W.: On a Cochrane sum and its hybrid mean value formula. J. Math. Anal. Appl., **267**, 89–96 (2002).

[5] Zhang, W., and Liu, H.: A Note on the Cochrane sum and its hybrid mean value formula. J. Math. Anal. Appl., **288**, 646–659 (2003).