Solutions of a pair of differential equations and their applications

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Abstract: In this paper, we consider the common solutions of a pair of differential equations and give some of their applications in the uniqueness problems of entire functions.

Key words: Differential equations; entire functions; entire solutions; uniqueness.

1. Introduction. In the study of the solutions of complex differential equations, the growth of a solution is a very important property. For linear differential equations of the form

(1) $$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = a(z),$$

where $a(z),\ a_0(z), \ldots, a_{n-1}(z)$ are polynomials, it is known that any entire solution of (1) must be of finite order, and if some of the coefficients $a_j(z)$ $(0 \le j \le n-1)$ are replaced by transcendental entire functions, then the equation (1) has at least one solution of infinite order. This can be proved by mainly using the Wiman-Valiron theory (see [3, 4, 6]).

It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory (see [2, 12]), and we say that two entire functions f and g share a finite value a CM (counting multiplicities), if $f-a$ and $g-a$ have the same zeros with the same multiplicities. In 1998, Gundersen [1] and Yang [1, 9] proved that every solutions of the differential equation

$$F^{(n)} - e^{\alpha(z)}F = 1$$

is of infinite order, where $\alpha(z)$ is a nonconstant entire functions. And so they proved.

Theorem A [1]. *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If the value a is shared by f, $f^{(n)}$, and $f^{(n+1)}$ CM, then $f \equiv f'$.*

In this paper, by using the Nevanlinna theory (see [2, 12]), we consider the common solutions of a pair of differential equations

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = a(z),$$

$$f^{(n+1)} + b_n(z)f^{(n)} + \cdots + b_0(z)f = b(z),$$

with some special entire coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$, and give some of their applications in the uniqueness theory of meromorphic functions.

2. Preliminary lemmas.

Lemma 1. *Let f be an entire function, $a \neq 0$ be a constant. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, and if there exists a constant $c \neq 0$ such that f satisfies one of the following differential equations:*

i) $f^{(n+1)} - a = c(f^{(n)} - a)$,
ii) $f^{(n)} - a = c(f - a)$,
iii) $f^{(n+1)} - a = c(f - a)$,
iv) $f^{(n+1)} = f^{(n)}$,

then $f \equiv f'$, and so $f = Ae^z$ for an arbitrary constant $A \neq 0$.

Proof. If f satisfies one of the differential equations i)–iv), then we know that f must be of finite order, [4]. By Theorem A, we have $f \equiv f'$. This completes the proof of Lemma 1. □

Lemma 2. *Let f be a common entire solution of a pair of differential equations*

$$\frac{f^{(n)} - 1}{f - 1} = e^{\alpha}, \quad \frac{f^{(n+1)} - 1}{f - 1} = e^{\beta},$$

where α and β $(\not\equiv \alpha)$ are nonconstant entire functions, then

$$T(r, e^{\alpha}) + T(r, e^{\beta}) = S(r, f).$$

Proof. From the conditions of Lemma 2, we know that f, $f^{(n)}$, and $f^{(n+1)}$ share 1 CM. Set

$$A(z) = \frac{f^{(n+1)} - f^{(n)}}{f - 1}, \quad B(z) = \frac{f^{(n+1)} - 1}{f^{(n)} - 1}.$$

Then $A(z) \not\equiv 0$ and $B(z)$ are entire functions and

$$T(r, A) = S(r, f).$$

By the second fundamental theorem of Nevanlinna

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theory, we have

$$T(r,B) \le N(r,B) + N\Big(r,\frac{1}{B}\Big) + N\Big(r,\frac{1}{B-1}\Big) + S(r,B) \le N\Big(r,\frac{1}{A}\Big) + S(r,B) = S(r,f).$$

Noticing that $e^{\alpha} = A/(B-1)$ and $e^{\beta} = Be^{\alpha}$, we get

$$T(r,e^{\alpha}) = S(r,f) \quad \text{and} \quad T(r,e^{\beta}) = S(r,f).$$

Lemma 2 is thus proved. □

By the same reasoning, we have

Lemma 3. *Let f be a common entire solution of a pair of differential equations*

$$\frac{f^{(n)}-1}{f-1} = e^{\alpha}, \quad \frac{f'-1}{f-1} = e^{\beta},$$

where α and β are nonconstant entire functions, then

$$T(r,e^{\alpha}) + T(r,e^{\beta}) = S(r,f).$$

Lemma 4 [7]. *Let $g(z)$ be a nonconstant meromorphic function, let n be a positive integer, and set*

$$P(z) = a_0 g^n + a_1 g^{n-1} + \cdots + a_{n-1} g + a_n,$$

where $a_0 \not\equiv 0$ and a_i $(i = 1, 2, \ldots, n)$ are meromorphic functions satisfying

$$T(r,a_i) = S(r,g), \quad i = 0, 1, \ldots, n.$$

Then

$$T(r,P) = nT(r,g) + S(r,g).$$

3. Solutions of a pair of differential equations.

Theorem 1. *Let $\alpha(z)$ and $\beta(z)$ are nonconstant entire functions such that $e^{\alpha(z)-\beta(z)} \not\equiv 1$. Then the following pair of differential equations*

$$f^{(n)} - e^{\alpha(z)} f = 1, \quad f^{(n+1)} - e^{\beta(z)} f = 1 \tag{2}$$

has no common solutions.

Proof. Suppose that the pair of equation (2) has a solution F, then $F(z)$ must be a nonconstant entire function which satisfies

$$F^{(n)} - e^{\alpha(z)} F = 1, \quad F^{(n+1)} - e^{\beta(z)} F = 1. \tag{3}$$

By differentiating the first equation and combining with the second equation of (3), we obtain

$$F' e^{\alpha} + F e^{\alpha} \alpha' = e^{\beta} F + 1.$$

Let $p = e^{-\alpha}$, $G = e^{\beta-\alpha} - \alpha'$. Then $F' = GF + p$ and

$$F'' = F(G^2 + G') + Gp + p'.$$

Now we assume that

$$F^{(k)} = F(G^k + H_{k-1}) + pH_{k-1} + p'H_{k-2} + \cdots + p^{(k-2)}H_1 + p^{(k-1)},$$

where H_j is a differential polynomial of G of degree j $(j \ge 1)$, then

$$F^{(k+1)} = F'(G^k + H_{k-1}) + F(kG^{k-1}G' + H'_{k-1}) + p'H_{k-1} + pH'_{k-1} + p''H_{k-2} + p'H'_{k-2} + p'''H_{k-3} + p''H'_{k-3} + \cdots + p^{(k)}.$$

Since the derivative of H_j $(j \ge 1)$ is also a differential polynomial of G of degree j (Here we denote by H_j a differential polynomial of degree j, which may not be the same each time it occurs), we obtain

$$F^{(k+1)} = (GF + p)(G^k + H_{k-1}) + F(kG^{k-1}G' + H'_{k-1}) + p'H_{k-1} + p''H_{k-2} + \cdots + p^{(k-1)}H_1 + p^{(k)}$$
$$= F(G^{k+1} + H_k) + pH_k + p'H_{k-1} + \cdots + p^{(k-1)}H_1 + p^{(k)}.$$

This proves by mathematical induction that, for any positive integer n,

$$F^{(n)} = F(G^n + H_{n-1}) + pH_{n-1} + p'H_{n-2} + \cdots + p^{(n-2)}H_1 + p^{(n-1)}, \tag{4}$$

where $p = e^{-\alpha}$, $G = e^{\beta-\alpha} - \alpha'$, and H_j is a differential polynomial of G of degree j $(j \ge 1)$.

From (3) and (4), we have

$$e^{\alpha} = G^n + H_{n-1} + \frac{1}{F}\{pH_{n-1} + p'H_{n-2} + \cdots + p^{(n-2)}H_1 + p^{(n-1)} - 1\}. \tag{5}$$

If

$$pH_{n-1} + p'H_{n-2} + \cdots + p^{(n-2)}H_1 + p^{(n-1)} \not\equiv 1,$$

then from (5), we have

$$T(r,F) \le nT(r,G) + \sum_{i=1}^{n-1} T(r,H_i) + \sum_{i=0}^{n-1} T(r,p^{(i)}) + O(1). \tag{6}$$

Set $f = F + 1$, then (3) becomes

$$\frac{f^{(n)} - 1}{f - 1} = e^{\alpha}, \quad \frac{f^{(n)} - 1}{f - 1} = e^{\beta}.$$

From Lemma 2, we have

$$T(r, e^{\alpha}) + T(r, e^{\beta}) = S(r, F). \tag{7}$$

Together with (6), we obtain

$$T(r, F) = S(r, F).$$

This is a contradiction.

If

$$pH_{n-1} + p'H_{n-2} + \cdots + p^{(n-2)}H_1 + p^{(n-1)} \equiv 1, \tag{8}$$

then

$$e^{\alpha} \equiv G^n + H_{n-1}. \tag{9}$$

By the definition of G and H_j, (9) gives

$$e^{\alpha} = (e^{\beta-\alpha})^n + l_1(e^{\beta-\alpha})^{n-1} + l_2(e^{\beta-\alpha})^{n-2} + \cdots + l_{n-1}e^{\beta-\alpha} + l_n, \tag{10}$$

where l_j $(j = 1, 2, \ldots, n)$ are polynomials of β, α and their derivatives.

If there exists an infinite set I with meas $I = \infty$, such that

$$T(r, e^{\beta-\alpha}) = o\{T(r, e^{\alpha})\}, \quad r \in I,$$

then from (9) and (10), we have

$$T(r, e^{\alpha}) \le nT(r, G) + T(r, H_{n-1}) + O(1) = o\{T(r, e^{\alpha})\}, \quad r \in I.$$

This is impossible. So there exists a set E with finite linear measure such that

$$T(r, e^{\alpha}) = O\{T(r, e^{\beta-\alpha})\}, \quad r \notin E,$$

and so

$$\sum_{i=1}^{n} T(r, l_i) = o\{T(r, e^{\beta-\alpha})\}, \quad r \notin E.$$

From Lemma 4 and (10), we obtain

$$T(r, e^{\alpha}) = nT(r, e^{\beta-\alpha}) + o\{T(r, e^{\beta-\alpha})\}, \quad r \notin E. \tag{11}$$

On the other hand, from (8) and $p = e^{-\alpha}$, we have

$$e^{\alpha} = H_{n-1} + \frac{p'}{p}H_{n-2} + \cdots + \frac{p^{(n-2)}}{p}H_1 + \frac{p^{(n-1)}}{p},$$

and so

$$T(r, e^{\alpha}) = (n-1)T(r, e^{\beta-\alpha}) + o\{T(r, e^{\beta-\alpha})\}, \quad r \notin E. \tag{12}$$

From (11) and (12), we have

$$T(r, e^{\beta-\alpha}) = o\{T(r, e^{\beta-\alpha})\}, \quad r \notin E.$$

This is also a contradiction. We complete the proof of Theorem 1. □

Theorem 2. *Let α be an entire function, and let β be a nonconstant entire function. Then the following pair of differential equations*

$$f^{(n)} - e^{\alpha}f = 1, \quad f' - e^{\beta}f = 1 \tag{13}$$

has no common solutions.

Proof. Suppose that the pair of equations (13) has a solution f. Then f is a transcendental entire function which satisfies

$$f^{(n)} - e^{\alpha}f = 1, \quad f' - e^{\beta}f = 1. \tag{14}$$

Set $f = F - 1$. Then F satisfies

$$\frac{F^{(n)} - 1}{F - 1} = e^{\alpha}, \quad \frac{F' - 1}{F - 1} = e^{\beta}. \tag{15}$$

From Lemma 3, we have

$$T(r, e^{\alpha}) + T(r, e^{\beta}) = S(r, F) = S(r, f). \tag{16}$$

Taking the derivatives on both sides of the second equation of (14) gives

$$f'' = f'e^{\beta} + fe^{\beta}\beta' = (fe^{\beta} + 1)e^{\beta} + fe^{\beta}\beta' = f(e^{2\beta} + e^{\beta}\beta') + e^{\beta}.$$

In the same manner, we have

$$f''' = f[e^{3\beta} + 3\beta' e^{2\beta} + e^{\beta}(\beta'\beta' + \beta'')] + e^{2\beta} + 2\beta' e^{\beta}.$$

By induction in number n, it can be easily obtained that

$$f^{(n)} = \{e^{n\beta} + p_1e^{(n-1)\beta} + p_2e^{(n-2)\beta} + \cdots + p_{n-1}e^{\beta} + p_n\}f + e^{(n-1)\beta} + \cdots + q_{n-2}e^{\beta} + q_{n-1} = Pf + Q, \tag{17}$$

where p_j $(j = 1, 2, \ldots, n-1)$ and q_j $(j = 1, 2, \ldots, n)$ are differential polynomials of β, and

$$P = e^{n\beta} + p_1e^{(n-1)\beta} + p_2e^{(n-2)\beta} + \cdots + p_{n-1}e^{\beta} + p_n,$$
$$Q = e^{(n-1)\beta} + q_1e^{(n-2)\beta} + \cdots + q_{n-2}e^{\beta} + q_{n-1}.$$

From (15), we know that f and $f^{(n)} - 1$ (i.e. $F - 1$

and $F^{(n)}-1$) have the same zeros with the same multiplicities, by W. K. Hayman's inequality [2, 12], we have

$$(18)\qquad T(r,f) \le \Big(2+\frac{1}{n}\Big)N\Big(r,\frac{1}{f}\Big)+S(r,f) + \Big(2+\frac{2}{n}\Big)\overline{N}\Big(r,\frac{1}{f^{(n)}-1}\Big) \le \Big(4+\frac{3}{n}\Big)\overline{N}\Big(r,\frac{1}{f}\Big)+S(r,f).$$

If $Q\equiv 1$, then from Lemma 4, we have

$$(n-1)T(r,e^{\beta})=S(r,e^{\beta}).$$

This contradicts that β is a nonconstant entire function. If $Q\not\equiv 1$, then from (16), (17), (18) and Lemma 4, it is easily seen that

$$T(r,f)\le 6\overline{N}\Big(r,\frac{1}{f}\Big)+S(r,f) \le 6N\Big(r,\frac{1}{Q-1}\Big)+S(r,f)=S(r,f).$$

This is also a contradiction. Theorem 2 is thus proved. □

Corollary. *If a pair of differential equations*

$$(19)\qquad f^{(n)}-e^{\alpha}f=1,\quad f'-e^{\beta}f=1$$

has a common nonconstant entire solution f, where α and β are entire functions and $n\,(\ge 2)$ is an integer, then α and β must be constants, and f assumes the form

$$f=ce^{Az}+1-1/A,$$

where $e^{\alpha}=e^{\beta}=A$, $A^{n-1}=1$, and c is a nonzero constant.

Proof. By Theorem 2, β must be a constant, and so f is of finite order. From the first equation of (19), we know that α is also a constant. Solving the (19), we can deduce the result of the Corollary. □

4. Applications. In 1986, Jank *et al.* proved the next two results.

Theorem B [5]**.** *Let f be a nonconstant meromorphic function, and let $a\ne 0$ be a finite constant. If f, f', and f'' share the value a CM, then $f\equiv f'$.*

Theorem C [5]**.** *Let f be a nonconstant entire function, and let $a\ne 0$ be a finite constant. If f and f' share the value a IM, and if $f''(z)=a$ whenever $f(z)=a$, then $f\equiv f'$.*

Theorem B suggests the following question of Yi-Yang.

Question 1 [11, p. 458]**.** Let f be a nonconstant meromorphic function, let $a\ne 0$ be a finite constant, and let n and m be positive integers satisfying $n<m$. If f, $f^{(n)}$, and $f^{(m)}$ share the value a CM, where n and m are not both even or both odd, must $f\equiv f^{(n)}$?

The following example [8] shows that the answer to Question 1 is, in general, negative. Let n and m be positive integers satisfying $m>n+1$, and let b be a constant which satisfies $b^n=b^m\ne 1$. Set $a=b^n$ and $f(z)=e^{bz}+a-1$. Then f, $f^{(n)}$, and $f^{(m)}$ share the value a CM, and $f\not\equiv f^{(n)}$.

Theorem A gives an affirmative answer to Question 1 in the case when f is an entire function of finite order and $m=n+1$.

Regarding Theorem A, a natural question is:

Question 2. What can be said when the function f in Theorem A is replaced by an entire function of infinite order [10]?

Theorem 3. *Let f be a nonconstant entire function, n be a positive integer. If f, $f^{(n)}$, and $f^{(n+1)}$ share a finite value $a\ne 0$ CM, then f must be of finite order.*

Proof. Suppose that there is a finite value $a\ne 0$ such that f, $f^{(n)}$, and $f^{(n+1)}$ share a CM, then there exist two entire functions α and β such that

$$\frac{f^{(n)}-a}{f-a}=e^{\alpha},\qquad \frac{f^{(n+1)}-a}{f-a}=e^{\beta}.$$

Let $F=f/a-1$. Then

$$T(r,f)=T(r,F)+O(1),$$

and F satisfies the following equations

$$F^{(n)}-e^{\alpha}F=1,$$
$$F^{(n+1)}-e^{\beta}F=1.$$

By Theorem 1, we have $e^{\alpha-\beta}\equiv 1$, or α and β are constants. In both cases, F is a solution of linear differential equation with constant coefficients, This implies that F must be of finite order, and so f is of finite order. Theorem 3 is proved. □

Remark. Theorem 3 gives an answer to the above Question 2, i.e. there is no infinite order entire functions that satisfy the condition of Theorem A. By the Corollary of Theorem 2, We have also the following result which is a complement of Theorem C.

Theorem 4. *Let f be a nonconstant entire function, n be a positive integer. If f, f', and $f^{(n)}$ share a finite value $a\ne 0$ CM, then f assumes the form*

$$f=ce^{Az}+a-a/A,$$

where $c \neq 0$ *and* A *are constants satisfying* $A^{n-1} = 1$.

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