Elliptic Hecke algebras and modified Cherednik algebras

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Abstract: The elliptic Hecke algebras associated to the 1-codimensional elliptic root systems have been defined by H. Yamada [10], which are subalgebras of Cherednik's double affine Hecke algebras [2, 3]. The elliptic Hecke algebras associated to the elliptic root systems of type $X^{(1,1)}$ have been defined similarly by the author [11] in terms of generators and relations associated to the completed elliptic diagram. On the other hand, M. Kapranov [6] has defined modified Cherednik algebras associated to the double coset decomposition of the group schemes over 2-dimensional local field. In this paper, we see that modified Cherednik algebras are isomorphic to elliptic Hecke algebras of type $X^{(1,1)}$.

Key words: Double affine Hecke algebras; elliptic Hecke algebras; modified Cherednik algebras.

1. Introduction. Let G be a Chevalley group over a $\mathfrak{p}$-adic field K associated to a complex semi-simple Lie algebra $\mathfrak{g}_{\mathbf{C}}$, and G' be the commutator subgroup of G. Let $B \subset G$ be a Borel subgroup and $B' = B \cap G'$, then N. Iwahori and H. Matsumoto [4] examined the structure of the double coset decomposition of G', G, with respect to B', B, respectively. The decompositions (so called the Bruhat decompositions) $G' = \bigcup_{\sigma \in \widetilde{W}'} B'w(\sigma)B'$ and $G = \bigcup_{\sigma \in \widetilde{W}} Bw(\sigma)B$ induce the structure of the affine Hecke algebra $\mathscr{H}(G', B')$, and the extended affine Hecke algebra $\mathscr{H}(G, B)$, where $\widetilde{W}'$ and $\widetilde{W}$ are the affine and the extended affine Weyl group, and we have $\widetilde{W} \cong \widetilde{W}' \rtimes \Pi$, by using a finite abelian group Π isomorphic to $P^\vee/Q^\vee$ (where $Q^\vee$ and $P^\vee$ are the coroot and coweight lattices of $\mathfrak{g}_{\mathbf{C}}$). The group Π acts on $\mathscr{H}(G', B')$ as a group of automorphism and $\mathscr{H}(G, B)$ is isomorphic to the "twisted" tensor product $\mathbf{Z}[\Pi] \otimes_{\mathbf{Z}} \mathscr{H}(G', B')$, with respect to this action. Recently, I. Cherednik defined "the double affine Hecke algebra" [2]. This is an algebra generated by three set of variables; T_i $(i = 1, \ldots, l)$, Y_λ $(\lambda \in P^\vee)$, X_μ $(\mu \in P)$, and the central element $q^{\pm 1/m}$, where Y_λ, T_i satisfy the relations of the extended affine Hecke algebra. In this construction, the generators Y_λ, T_i $(i = 1, \ldots, l)$ are replaced with Π, $T_0, \ldots, T_l$ which generate the same extended affine Hecke algebra, and the subalgebra generated by $T_1, \ldots, T_l$, X_μ $(\mu \in P)$ satisfy the relations of the extended affine Hecke algebra for the root system $R^\vee$ (where $R^\vee$ is the dual root system of R) (see A. Kirillov [9]). But the double affine Hecke algebra is also differently defined by the generators $T_0, \ldots, T_l$, Π, X_μ $(\mu \in P^\vee)$ and $q^{\pm 1/m}$ [3]. In this case $Q^\vee \subset P^\vee$ and in the previous case, by considering the embedding of lattices $Q^\vee \hookrightarrow P$, we can consider the subalgebra generated by the elements T_i $(0 \leq i \leq l)$, X_β $(\beta \in Q^\vee)$ and $q^{\pm 1}$. We will see that this subalgebra is isomorphic to the elliptic Hecke algebra of type $X^{(1,1)}$ defined by the author in [11]. Similarly to the case of the $\mathfrak{p}$-adic field (i.e., 1-dimensional local field), in the case of 2-dimensional local field K, for the group scheme $G(K)$, one can consider the problem to decompose $G(K)$ to the double coset spaces with respect to a Borel subgroup (see A. N. Parshin [8]), and to describe the associated Hecke algebra. M. Kapranov [6] has given one answer to this problem, and constructed the modified Cherednik algebra $\mathscr{H}(\Gamma, \Delta_1)$ which is a subalgebra of the double affine Hecke algebra. In this article, we will show that $\mathscr{H}(\Gamma, \Delta_1)$ is isomorphic to the elliptic Hecke algebra of type $X^{(1,1)}$.

2. Double affine Hecke algebras and elliptic Hecke algebras. Let R be a root system of type X $(X = A_l, B_l, \ldots, G_2)$, and $Q^\vee$, $P^\vee$ be the coroot lattice, the coweight lattice of R. Let $\widetilde{R} := R \times \mathbf{Z}$ and $\widehat{R} := R \times \mathbf{Z} \times \mathbf{Z}$ be the affine root system of type $X^{(1)}$ and the elliptic root system of type $X^{(1,1)}$ (see [1]), respectively. Let W be the Weyl group as-

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sociated to R, then the elliptic Weyl group and the extended elliptic Weyl group of type $X^{(1,1)}$ are realized by the semi-direct product $W \ltimes (Q^\vee \times Q^\vee)$ and $W \ltimes (P^\vee \times P^\vee)$, respectively. The quotient group $P^\vee/Q^\vee \cong \Pi$ acts on the system of simple roots of the affine root system $\widetilde{R}$ by permutations. Now let us recall the definiton of the double affine Hecke algebras [3]. Let $\mathbf{C}_{q,t}$ be the field of rational functions in terms of independent variables $q^{1/m}$, $\{t_j^{1/2} := t_{\alpha_j}^{1/2}$ $(0 \le j \le l)\}$, where $m = 2$ for D_{2k} and C_{2k+1}, $m = 1$ for C_{2k}, B_l, otherwise $m = |\,\Pi\,|$. Let $\alpha_1, \dots, \alpha_l$ be the basis of simple roots in R, and $\alpha_0 = -\theta + \delta$, $\alpha_1, \dots, \alpha_l$ be the basis of simple roots in $\widetilde{R}$, where $\theta \in R$ is the maximal root.

Definition 2.1 (I. Cherednik [3]). The double affine Hecke algebra $\mathscr{H}$ is generated over the field $\mathbf{C}_{q,t}$ by the elements $\{T_j,\ 0 \le j \le l\}$, pairweise commutative $\{X_{\beta^\vee},\ \beta^\vee \in P^\vee\}$ $(\beta^\vee := 2\beta/\langle\beta,\beta\rangle)$, group Π and the central element $q^{\pm 1/m}$. Let $X_{\beta^\vee + k\delta} := X_{\beta^\vee} q^k$ for $\beta^\vee \in P^\vee$, $k \in (1/m)\mathbf{Z}$. Then the following relations are imposed.

$$\begin{cases}
\text{(0)} & (T_j - t_j^{1/2})(T_j + t_j^{-1/2}) = 0, \quad 0 \le j \le l, \\
\text{(i)} & T_i T_j T_i \cdots = T_j T_i T_j \cdots, \\
 & \qquad m_{ij} \text{ factors on each side,} \\
 & (m_{ij} = 2,3,4,6 \text{ if } \alpha_i \text{ and } \alpha_j \text{ are joined} \\
 & \qquad \text{by } 0,1,2,3 \text{ laces respectively}), \\
\text{(ii)} & \pi_r T_i \pi_r^{-1} = T_j \text{ if } \pi_r(\alpha_i) = \alpha_j, \\
\text{(iii)} & T_i X_{\beta^\vee} T_i = X_{\beta^\vee - \alpha_i^\vee} \\
 & \qquad \text{if } \langle\beta^\vee, \alpha_i\rangle = 1,\ 1 \le i \le l, \\
\text{(iv)} & T_0 X_{\beta^\vee} T_0 = X_{s_0(\beta^\vee)} \text{ if } \langle\beta^\vee, \theta\rangle = -1, \\
\text{(v)} & T_i X_{\beta^\vee} = X_{\beta^\vee} T_i \\
 & \qquad \text{if } \langle\beta^\vee, \alpha_i\rangle = 0 \text{ for } 0 \le i \le l, \\
\text{(vi)} & \pi_r X_{\beta^\vee} \pi_r^{-1} = X_{\pi_r(\beta^\vee)}.
\end{cases}$$

Let us introduce the element $X_{\alpha_0^\vee} := X_{\alpha_1^\vee}^{-n_1} \cdots X_{\alpha_l^\vee}^{-n_l} q$ for $\alpha_0^\vee := -n_1\alpha_1^\vee - \cdots - n_l\alpha_l^\vee + \delta$, and define the algebra $\mathscr{H}_{el}$ which is a subalgebra of the double affine Hecke algebra $\mathscr{H}$ as follows:

Definition 2.2. Let $\mathbf{C}_t$ be the field of rational functions of the variables $t_j^{1/2} = t_{\alpha_j}^{1/2}$ $(0 \le j \le l)$, then we define the algebra $\mathscr{H}_{el}$ by the following set of generators and relations.

Generators: T_α for $\alpha \in \{\alpha_0, \dots, \alpha_l\}$, $X_{\alpha^\vee}$ for $\alpha^\vee \in Q^\vee$ and $q^{\pm 1}$.

Relations: $X_{\alpha^\vee} X_{\beta^\vee} = X_{\beta^\vee} X_{\alpha^\vee}$ for $\alpha^\vee$, $\beta^\vee \in Q^\vee$ and

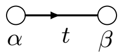
$$\begin{cases}
\text{(0)} & (T_\alpha - t_\alpha^{1/2})(T_\alpha + t_\alpha^{-1/2}) = 0, \\
\text{(i)} & T_\alpha T_\gamma T_\alpha \cdots = T_\gamma T_\alpha T_\gamma \cdots, \\
 & \qquad m_{\alpha\gamma} \text{ factors on each side,} \\
 & (m_{\alpha\gamma} = 2,3,4,6 \text{ if } \alpha \text{ and } \gamma \text{ are joined} \\
 & \qquad \text{by } 0,1,2,3 \text{ laces respectively}), \\
\text{(ii)} & T_\alpha X_{-\beta^\vee} T_\alpha = X_{-\beta^\vee - \alpha^\vee} \text{ if } \langle\beta^\vee, \alpha\rangle = -1, \\
 & T_\alpha X_{-\beta^\vee} = X_{-\beta^\vee} T_\alpha \text{ if } \langle\beta^\vee, \alpha\rangle = 0.
\end{cases}$$

Remark 2.3. The inner product $\langle\cdot,\cdot\rangle$ is normalized by $\langle\alpha,\alpha\rangle = 2$ for long roots α, and this induces that $\langle\alpha_0,\alpha_0\rangle = 2$ for all root system R.

Remark 2.4. By the following reformulation,

$$\begin{aligned}
& T_{\alpha_0} X_{\beta^\vee} T_{\alpha_0} = X_{s_0(\beta^\vee)} && \text{if } \langle\beta^\vee, \theta\rangle = -1 \\
\Leftrightarrow\ & T_{\alpha_0} X_{\beta^\vee} T_{\alpha_0} = X_{s_0(\beta^\vee)} && \text{if } \langle\beta^\vee, \alpha_0\rangle = 1 \\
\Leftrightarrow\ & T_{\alpha_0} X_{\beta^\vee} T_{\alpha_0} = X_{\beta^\vee - \alpha_0^\vee} && \text{if } \langle\beta^\vee, \alpha_0\rangle = 1 \\
\Leftrightarrow\ & T_{\alpha_0} X_{-\beta^\vee} T_{\alpha_0} = X_{-\beta^\vee - \alpha_0^\vee} && \text{if } \langle\beta^\vee, \alpha_0\rangle = -1
\end{aligned}$$

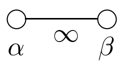
the relations (iii) and (iv) in $\mathscr{H}$ are reduced to the first relation of (ii) in $\mathscr{H}_{el}$.

Remark 2.5. By the inner products $\langle\alpha^\vee, \beta\rangle = -1$, $\langle\alpha, \beta^\vee\rangle = -t$ for the diagram 

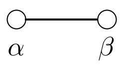
the relations (0), (i) and (ii) in $\mathscr{H}_{el}$ are easily described in terms of the Dynkin diagram as follows:

$\Longrightarrow$ $(T_\alpha - t_\alpha^{1/2})(T_\alpha + t_\alpha^{-1/2}) = 0.$

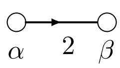
$\Longrightarrow$ $T_\alpha T_\beta = T_\beta T_\alpha,$ $T_\alpha X_{-\beta^\vee} = X_{-\beta^\vee} T_\alpha,$ $T_\beta X_{-\alpha^\vee} = X_{-\alpha^\vee} T_\beta.$



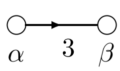
$\Longrightarrow$ $T_\alpha X_{-\alpha^\vee - \beta^\vee} = X_{-\alpha^\vee - \beta^\vee} T_\alpha,$ $T_\beta X_{-\alpha^\vee - \beta^\vee} = X_{-\alpha^\vee - \beta^\vee} T_\beta.$



$\Longrightarrow$ $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta,$ $T_\alpha X_{-\beta^\vee} T_\alpha = X_{-\beta^\vee - \alpha^\vee},$ $T_\beta X_{-\alpha^\vee} T_\beta = X_{-\alpha^\vee - \beta^\vee}.$



$\Longrightarrow$ $(T_\alpha T_\beta)^2 = (T_\beta T_\alpha)^2,$ $T_\alpha X_{-\alpha^\vee - \beta^\vee} = X_{-\alpha^\vee - \beta^\vee} T_\alpha,$ $T_\beta X_{-\alpha^\vee} T_\beta = X_{-\alpha^\vee - \beta^\vee}.$



$\Longrightarrow$ $(T_\alpha T_\beta)^3 = (T_\beta T_\alpha)^3,$ $T_\alpha X_{-\alpha^\vee - \beta^\vee} T_\alpha = X_{-2\alpha^\vee - \beta^\vee},$ $T_\beta X_{-\alpha^\vee} T_\beta = X_{-\alpha^\vee - \beta^\vee}.$

Here we set $T_\alpha^* := T_{\alpha^*} := T_\alpha^{-1} X_{-\alpha^\vee}$, and a, b denote one of the elements $\{\alpha, \alpha^*\}$, $\{\beta, \beta^*\}$ respectively, then we obtain the following.

Proposition 2.6. *The algebra $\mathscr{H}_{el}$ is described by the following set of generators and relations*:

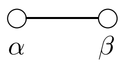
Generators: T_α, T_{α^*} *for* $\alpha \in \{\alpha_0, \dots, \alpha_l\}$.

Relations:

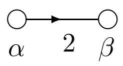
(I)

$$\Longrightarrow \quad (T_a - t_\alpha^{1/2})(T_a + t_\alpha^{-1/2}) = 0, \quad (t_{\alpha^*} = t_\alpha).$$

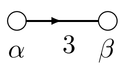
$$\Longrightarrow \quad T_a T_b = T_b T_a.$$



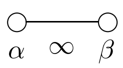
$$\Longrightarrow \quad \begin{aligned} &T_a T_b T_a = T_b T_a T_b, \\ &T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha, \\ &T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta, \\ &T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^*. \end{aligned}$$



$$\Longrightarrow \quad \begin{aligned} &(T_a T_b)^2 = (T_b T_a)^2, \\ &T_\alpha^* T_\beta T_\beta^* T_\alpha = T_\beta T_\beta^* T_\alpha T_\alpha^*, \\ &T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta, \\ &T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^*. \end{aligned}$$



$$\Longrightarrow \quad \begin{aligned} &(T_a T_b)^3 = (T_b T_a)^3, \\ &T_\alpha^* T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\alpha T_\alpha^* T_\beta T_\beta^* T_\alpha, \\ &T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta, \\ &T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^*. \end{aligned}$$



$$\Longrightarrow \quad \begin{aligned} T_\alpha T_\alpha^* T_\beta T_\beta^* &= T_\alpha^* T_\beta T_\beta^* T_\alpha \\ &= T_\beta T_\beta^* T_\alpha T_\alpha^* = T_\beta^* T_\alpha T_\alpha^* T_\beta. \end{aligned}$$

(II)

$$A_l^{(1,1)} \ (l \geq 1) \Longrightarrow T_0 T_0^* T_1 T_1^* \cdots T_l T_l^* = q^{-1},$$

$$B_l^{(1,1)} \ (l \geq 3) \Longrightarrow T_0 T_0^* T_1 T_1^* (T_2 T_2^* \cdots T_{l-1} T_{l-1}^*)^2 T_l T_l^* = q^{-1},$$

$$C_l^{(1,1)} \ (l \geq 2) \Longrightarrow T_0 T_0^* T_1 T_1^* \cdots T_l T_l^* = q^{-1},$$

$$D_l^{(1,1)} \ (l \geq 4) \Longrightarrow T_0 T_0^* T_1 T_1^* (T_2 T_2^* \cdots T_{l-2} T_{l-2}^*)^2 T_{l-1} T_{l-1}^* T_l T_l^* = q^{-1},$$

$$E_6^{(1,1)} \Longrightarrow T_0 T_0^* T_1 T_1^* (T_2 T_2^*)^2 (T_3 T_3^*)^3 (T_4 T_4^*)^2 T_5 T_5^* (T_6 T_6^*)^2 = q^{-1},$$

$$E_7^{(1,1)} \Longrightarrow T_0 T_0^* T_1 T_1^* (T_2 T_2^*)^2 (T_3 T_3^*)^3 (T_4 T_4^*)^4 (T_5 T_5^*)^3 (T_6 T_6^*)^2 (T_7 T_7^*)^2 = q^{-1},$$

$$E_8^{(1,1)} \Longrightarrow T_0 T_0^* (T_1 T_1^*)^2 (T_2 T_2^*)^3 (T_3 T_3^*)^2 T_4 T_4^* = q^{-1},$$

$$F_4^{(1,1)} \Longrightarrow T_0 T_0^* (T_1 T_1^*)^2 (T_2 T_2^*)^3 (T_3 T_3^*)^2 T_4 T_4^* = q^{-1},$$

$$G_2^{(1,1)} \Longrightarrow T_0 T_0^* (T_1 T_1^*)^2 T_2 T_2^* = q^{-1}.$$

Proof. From $X_{-\alpha^\vee} = T_\alpha T_\alpha^*$, we obtain the following relations:

$$\begin{aligned} X_{\alpha^\vee} X_{\beta^\vee} &= X_{\beta^\vee} X_{\alpha^\vee} \\ &\Longrightarrow \quad T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^*, \\ T_\alpha X_{-\alpha^\vee - \beta^\vee} &= X_{-\alpha^\vee - \beta^\vee} T_\alpha \\ &\Longrightarrow \quad T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\alpha^* T_\beta T_\beta^* T_\alpha, \\ T_\alpha X_{-\beta^\vee} T_\alpha &= X_{-\alpha^\vee - \beta^\vee} \\ &\Longrightarrow \quad T_\beta T_\beta^* T_\alpha = T_\alpha^* T_\beta T_\beta^*, \end{aligned}$$

and from $X_{\alpha_0^\vee} = X_{\alpha_1^\vee}^{-n_1} \cdots X_{\alpha_l^\vee}^{-n_l} q$, we obtain

$$T_0 T_0^* (T_1 T_1^*)^{n_1} (T_2 T_2^*)^{n_2} \cdots (T_l T_l^*)^{n_l} = q^{-1}.$$

Further, in the next cases, from the relations of the left hand side, we can obtain the relations of the right hand side, which has been already proved in [11] (in the proof of Proposition 4.2).

$$\begin{cases} T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta \\ T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha \\ T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta \\ T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^* \end{cases} \Rightarrow \begin{cases} T_\alpha T_\beta^* T_\alpha = T_\beta^* T_\alpha T_\beta^* \\ T_\beta T_\alpha^* T_\beta = T_\alpha^* T_\beta T_\alpha^* \\ T_\alpha^* T_\beta^* T_\alpha^* = T_\beta^* T_\alpha^* T_\beta^* \end{cases}$$

$$\begin{cases} (T_\alpha T_\beta)^2 = (T_\beta T_\alpha)^2 \\ T_\alpha^* T_\beta T_\beta^* T_\alpha = T_\alpha T_\alpha^* T_\beta T_\beta^* \\ T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta \\ T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^* \end{cases} \Rightarrow \begin{cases} (T_\alpha T_\beta^*)^2 = (T_\beta^* T_\alpha)^2 \\ (T_\beta T_\alpha^*)^2 = (T_\alpha^* T_\beta)^2 \\ (T_\alpha^* T_\beta^*)^2 = (T_\beta^* T_\alpha^*)^2 \end{cases}$$

$$\begin{cases} (T_\alpha T_\beta)^3 = (T_\beta T_\alpha)^3 \\ T_\alpha^* T_\alpha T_\alpha^* T_\beta T_\beta^* \\ \qquad = T_\alpha T_\alpha^* T_\beta T_\beta^* T_\alpha \\ T_\beta^* T_\alpha T_\alpha^* = T_\alpha T_\alpha^* T_\beta \\ T_\alpha T_\alpha^* T_\beta T_\beta^* = T_\beta T_\beta^* T_\alpha T_\alpha^* \end{cases} \Rightarrow \begin{cases} (T_\alpha T_\beta^*)^3 = (T_\beta^* T_\alpha)^3 \\ (T_\beta T_\alpha^*)^3 = (T_\alpha^* T_\beta)^3 \\ (T_\alpha^* T_\beta^*)^3 = (T_\beta^* T_\alpha^*)^3 \end{cases}$$

so the proof is completed. □

Remark 2.7. From Proposition 2.6, we see that the algebra $\mathscr{H}_{el}$ is isomorphic to the elliptic Hecke algebra of type $X^{(1,1)}$ defined in [11].

3. Modified Cherednik algebras. Let us recall the results in [5, 6] and [7]. Let G be a split simple, simply-connected algebraic group (over $\mathbf{Z}$), $T \subset G$ the fixed maximal torus, and we regard G, T as group schemes. Let $L = \mathrm{Hom}(\mathbf{G}_m, T)$ and $L^\vee = \mathrm{Hom}(T, \mathbf{G}_m)$ be the coweight and weight lattices of G, $R \subset L^\vee$ be the root system. Let $T^\vee = \mathrm{Spec}\,\mathbf{C}[L]$ be the complex torus dual to T. Let $L_{\mathit{aff}} = \mathbf{Z} \oplus L$ be the lattice of affine coweight of G. Let W and $W_{\mathit{aff}} := W \ltimes L$ be the Weyl group and the affine Weyl group of G. Let $W_{el} := W \ltimes (L \oplus L)$ be the elliptic Weyl group (double affine Weyl group) and $\widetilde{W} := W_{\mathit{aff}} \ltimes L_{\mathit{aff}}$ be its central extension (double affine Heisenberg-Weyl group). Let $T^\vee_{\mathit{aff}} = \mathrm{Spec}\,\mathbf{C}[L_{\mathit{aff}}]$ be the affine torus corresponding to $T^\vee$. Here we note that as G is simply connected, in the notation of the previous section, we can identify $L = Q^\vee$, $L^\vee = P$. Set $P_{\mathit{aff}} = P \oplus (1/m)\mathbf{Z}$, $\widetilde{T}_{\mathit{aff}} = \mathrm{Spec}\,\mathbf{C}[P_{\mathit{aff}}]$, where $m \in \mathbf{Z}_+$ is the smallest integer such that $m\langle \lambda, \mu \rangle \in \mathbf{Z}$ for every $\lambda \in P^\vee$, $\mu \in P$. Let $\mathbf{C}(T^\vee_{\mathit{aff}})$ and $\mathbf{C}(\widetilde{T}_{\mathit{aff}})$ be the field of rational functions on $T^\vee_{\mathit{aff}}$ and $\widetilde{T}_{\mathit{aff}}$, respectively, then the double affine Hecke algebra $\mathscr{H}$ is realized by the subalgebra consisting of finite linear combinations $\sum_{w \in W \ltimes P^\vee} f_w(t)[w]$ with $f_w(t) \in \mathbf{C}(\widetilde{T}_{\mathit{aff}})$ satisfying certain residue conditions (see [6]). Classically, for a locally compact group G and its compact subgroup Δ, the Hecke algebra $\mathscr{H}(G, \Delta)$ can be defined as the algebra compactly supported double Δ-invariant continuous functions of G with the operation given by the convolution with respect to the Haar measure on G. In the case of $G(K)$ with 2-dimensional local field K, for that purpose, M. Kapranov defined the Hecke algebra $\mathscr{H}(\Gamma, \Delta_1)$, for the central extension Γ of $G(K)$ and an appropriate subgroup $\Delta_1 \subset \Gamma$. Further he showed that $\mathscr{H}(\Gamma, \Delta_1)$ is a subalgebra of the double affine Hecke algebra $\mathscr{H}$ consisting of linear combinations as above but going over $W \ltimes Q^\vee \subset W \ltimes P^\vee$ with $f_w(t) \in \mathbf{C}(T^\vee_{\mathit{aff}})$, and called $\mathscr{H}(\Gamma, \Delta_1)$ "the modified Cherednik algebra". From these arguments, we have the following.

Proposition 3.1. *The modified Cherednik algebra $\mathscr{H}(\Gamma, \Delta_1)$ is isomorphic to the elliptic Hecke algebra of type $X^{(1,1)}$.*

Proof. We use the definition ([2, 6]) of the Cherednik algebra $\mathscr{H}_r$ with generators

$$Y_{(b,n)} := Y_b\, q^n, \quad (b, n) \in P_{\mathit{aff}} = P \oplus \frac{1}{m}\mathbf{Z},$$
$$\tau_w, \ w \in \widehat{W} := W \ltimes Q^\vee, \quad \text{and} \quad \tau_\pi, \ \pi \in \Pi.$$

From Remark 2.7, we see that the elliptic Hecke algebra of type $X^{(1,1)}$ is isomorphic to the subalgebra of $\mathscr{H}_r$ generated by $Y_{(b,n)}$ for $(b,n) \in Q^\vee \oplus \mathbf{Z}$ and τ_i $(0 \le i \le l)$ with the relations in Definition 2.2 by the correspondence $Y_{(b,n)} \leftrightarrow X_b q^n$, $\tau_i \leftrightarrow T_i$. Here we note that $\{\tau_w,\ w \in W \ltimes Q^\vee\} \cong \{\tau_0, \tau_1, \ldots, \tau_l\ (\tau_i := \tau_{s_i})\}$. From the results in [6]

$$\mathbf{C}(\widetilde{T}^{\mathit{aff}})[W \ltimes P^\vee] \cong \mathbf{C}(\widetilde{T}^{\mathit{aff}})[W \ltimes Q^\vee][\breve{\Pi}],$$

and

$$\mathscr{H}_r \cong \big\{\text{the subalgebra in } \mathbf{C}(\widetilde{T}^{\mathit{aff}})[W \ltimes P^\vee] \text{ consisting of } \textstyle\sum f_w(\lambda)[w] \text{ such that } f_w \text{ satisfy the certain residue conditions ([6])}\big\}.$$

The correspondence of the generators of the both algebras has been given by ([5, 6]);

$$\begin{aligned} Y_{(b,n)} &\leftrightarrow t_{(b,n)} := t^b \zeta^n \\ \tau_i &\leftrightarrow \sigma_i := \Big(\frac{\zeta t^{\alpha_i} - \zeta^{-1}}{t^{\alpha_i} - 1}\Big)[s_i] - \frac{\zeta - \zeta^{-1}}{t^{\alpha_i} - 1}[1] \\ \tau_\pi &\leftrightarrow [\pi]. \end{aligned}$$

The modified Cherednik algebra $\mathscr{H}(\Gamma, \Delta_1)$ is the subalgebra in $\mathbf{C}(T^\vee_{\mathit{aff}})[W \ltimes Q^\vee]$ consisting of $\sum f_w(\lambda)[w]$ such that f_w satisfy the same residue conditions as $\mathscr{H}_r$, and owing to the result [6] (Theorem 3.3.8), which is isomorphic to $\mathbf{C}(T^\vee_{\mathit{aff}})[W \ltimes Q^\vee] \cap \mathscr{H}_r$. Therefore we see that $\mathscr{H}(\Gamma, \Delta_1)$ is the algebra generated by the elements

$$t_{(b,n)} = t^b \zeta^n, \quad (b,n) \in Q^\vee_{\mathit{aff}} = Q^\vee \oplus \mathbf{Z} \quad \text{and} \quad \sigma_i \ (i = 0, \ldots, l),$$

with the relations in Definition 2.2, and which completes the proof. □

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