Real spectrum with Nash structural sheaf

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Abstract: We show that the Nash structural sheaf over the real spectrum of a commutative ring is determined only by the underlying space. We also calculate the stalks and global sections of it in the restricted case. As an application, we show some basic properties of 'separated' morphisms.

Key words: Real spectrum; Nash structural sheaf.

1. Introduction. Roy defined the following Nash structural sheaf over the real spectrum of a commutative ring [7].

Definition 1.1 ($\mathcal{N}_{\mathrm{Spec}_r A}$)**.** Let A be a commutative ring, and U a constructible open subset of $\mathrm{Spec}_r A$. Let $I(U)$ be the inductive system whose elements are (B, f, s) with $f : A \longrightarrow B$ an étale A-algebra and s a section from the local homeomorphism $\mathrm{Spec}_r f$ over U, whose morphisms from (B, f, s) to (B', f', s') are A-algebra morphisms $g : B \longrightarrow B'$ with $g \circ f = f'$ and $s = \mathrm{Spec}_r g \circ s'$ [6, pp. 13–14].

The inductive limit $\mathcal{A}(U)$ of $I(U)$ defines a presheaf which is separated [1, Proposition 4.2.2]. The Nash structural sheaf over $\mathrm{Spec}_r A$ noted $\mathcal{N}_{\mathrm{Spec}_r A}$ is the sheaf associated to $\mathcal{A}$.

Let R be a real closed field and $M \subset R^n$ be a Nash submanifold. For any open semialgebraic subset U of M, $\mathcal{N}(U)$ denotes the ring of Nash functions on U. The rings $\mathcal{N}(U)$ form a sheaf $\tilde{\mathcal{N}}_{\tilde{M}}$ for the Grothendieck topology on the lattice of open semialgebraic subsets of M which is generated by the finite open semialgebraic covering. Here $\tilde{M}$ denotes the topological space whose underlying space M with the above semialgebraic topology. Remember that a natural homomorphism $R[X_1, \ldots, X_n] \to \mathcal{N}(M)$ induces an homeomorphism $\mathrm{Spec}_r \mathcal{N}(M) \to \tilde{M}$. Furthermore, the sheaf $\tilde{\mathcal{N}}_{\tilde{M}}$ coincides with the sheaf $\mathcal{N}_{\mathrm{Spec}_r \mathcal{N}(M)}$ via the above homeomorphism.

We consider the following ringed space.

Definition 1.2. An affine real scheme is a locally ringed space $(\mathrm{Spec}_r A, \mathcal{N}_{\mathrm{Spec}_r A})$, where A is a ring.

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A real scheme is a locally ringed space $(X, \mathcal{N}_X)$ which satisfies the following condition.

There exists an open covering $\{U_i\}_{i \in I}$ of X such that, for any $i \in I$, there exists a ring A which satisfies

$$(U_i, \mathcal{N}_X|U_i) \simeq (\mathrm{Spec}_r A, \mathcal{N}_{\mathrm{Spec}_r A})$$

as a ringed space. We call this covering an affine open covering.

In the present paper, we give easy descriptions of a stalk of this Nash sheaf over a real spectrum and sections of this sheaf over an open basis. We also show that the Nash structural sheaf $\mathcal{N}_X$ is determined only by the underlying topological space X.

We define a *separated morphism of real schemes* in Section 3. As an application, we investigate this morphism using our result.

2. General facts. A *real closed local ring* is an henselian ring with real closed residue field. The real spectrum of a real closed local ring A has a unique maximal prime cone μ_A [1, Proposition 2.3.1]. The real strict localization of a ring is defined as follows in [1].

Definition 2.1 (A_α)**.** Let A be a ring and α a prime cone of A. A real strict localization of A at α is an A-algebra $i : A \longrightarrow A'$ such that:

1. A' is a real closed local ring with maximal prime cone ν_A, and the statement $i^{-1}(\mu_{A'}) = \alpha$ satisfies.
2. (universal property) for any closed local A-algebra $f : A \longrightarrow B$ such that, μ_B being the maximal prime cone of B such that $f^{-1}(\mu_B) = \alpha$, there exists a unique local A-algebra morphism g from A' to B such that $g \circ i = f$

Then there exists a unique (up to isomorphism) real strict localization of a ring at a prime cone

Proposition 2.2. *Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be two real schemes such that there exists a morphism $f : (X, \mathcal{N}_X) \longrightarrow (Y, \mathcal{N}_Y)$ and f is a homeomorphism from a topological space X to a topological space Y.*

Then $(X, \mathcal{N}_X)$ is isomorphic to $(Y, \mathcal{N}_Y)$.

Proof. Since the family of open affine sets is an open basis of X, we have only to prove that $\mathcal{N}_{\mathrm{Spec}_r B}|_{\mathrm{Spec}_r A} \cong \mathcal{N}_{\mathrm{Spec}_r A}$ in the case when there exists an inclusion $\mathrm{Spec}_r A \hookrightarrow \mathrm{Spec}_r B$ and a homomorphism $B \longrightarrow A$. In short, we have only to prove that $\mathcal{N}_{\mathrm{Spec}_r B, \alpha} \cong \mathcal{N}_{\mathrm{Spec}_r A, \alpha}$, where $\alpha \in \mathrm{Spec}_r A$.

By definition of A_α, there exists a unique homomorphism $B_\alpha \longrightarrow A_\alpha$.

On the other hand, there exists an affine open set $\mathrm{Spec}_r B'$ of $\mathrm{Spec}_r A$ such that $\alpha \in \mathrm{Spec}_r B' \hookrightarrow \mathrm{Spec}_r A \hookrightarrow \mathrm{Spec}_r B$ by [2, Theorem 23] because a family of constructive open subsets is an open basis of $\mathrm{Spec}_r B$. In this case, there exists a homomorphism $\Gamma(\mathrm{Spec}_r A, \mathcal{N}_{\mathrm{Spec}_r A}) \longrightarrow \Gamma(\mathrm{Spec}_r B', \mathcal{N}_{\mathrm{Spec}_r B'})$. Hence there exists a morphism $(\mathrm{Spec}_r B', \mathcal{N}_{\mathrm{Spec}_r B'}) \longrightarrow (\mathrm{Spec}_r A, \mathcal{N}_{\mathrm{Spec}_r A})$ by [1, Theorem 4.2.4]. As above, there exists a morphism $B_\alpha \longrightarrow A_\alpha \longrightarrow B'_\alpha = B_\alpha$. Thus $A_\alpha \cong B_\alpha$ because there exist no homomorphisms $B_\alpha \longrightarrow B_\alpha$ but an identity. i.e., $\mathcal{N}_{\mathrm{Spec}_r A, \alpha} \cong \mathcal{N}_{\mathrm{Spec}_r B, \alpha}$. □

Definition 2.3. A ring N satisfies an idempotency property if this ring N satisfies $N = \Gamma(\mathrm{Spec}_r N, \mathcal{N}_{\mathrm{Spec}_r N})$. The ring $N = \Gamma(\mathrm{Spec}_r A, \mathcal{N}_{\mathrm{Spec}_r A})$ satisfies an idempotency property by [2, Theorem 23].

Proposition 2.4. *Let N be a ring which satisfies an idempotency property.*

(a) *For any $\alpha \in \mathrm{Spec}_r N$, $N_\alpha = \mathcal{N}_{\mathrm{Spec}_r N, \alpha} = N_{\mathrm{supp}(\alpha)}$.*

(b) *Set $\mathcal{U}(f_1, \dots, f_p) := \{s \in M; f_1(x) > 0, \dots, f_p(x) > 0\}$ for any elements $f_1, \dots f_p \in N$. Then*

$$\mathcal{N}(\mathcal{U}(f_1, \dots, f_p)) = (\cdots (N_{f_1})_{f_2} \cdots)_{f_p}.$$

We will abbreviate this ring to $N_{f_1, \dots, f_p}$ later.

Proof. (a): We have only to show that the ring $N_{\mathrm{supp}(\alpha)}$ satisfies the condition of Definition 2.1.

(i) We first prove that $N_{\mathrm{supp}(\alpha)}$ is a real closed local ring. It is obvious that $N_{\mathrm{supp}(\alpha)}$ is local.

Now recall that $k(\mathrm{supp}(\alpha))$ denotes the residue field of $N_{\mathrm{supp}(\alpha)}$ and $k(\alpha)$ denotes the real closure of $k(\mathrm{supp}(\alpha))$. For any $x \in k(\alpha)$, there exists a polynomial P with coefficients in $k(\mathrm{supp}(\alpha))$ such that x is a simple root of P. Let Q be a polynomial with coefficients in $N_{\mathrm{supp}(\alpha)}$ such that $\bar{Q} = P$. We can assume that $Q \in N[X]$.
Let m_Q be the kernel of the homomorphism

$$B = N_{\mathrm{supp}(\alpha)}[X]/(Q) \ni X \mapsto x \in k(\alpha).$$

Let R be an element of B such that $R \notin m_Q$ and Q' is invertible in B_R (it exists because $Q' \notin m_Q$). In the same way, there exists an element h of N such that $h \notin \mathrm{supp}(\alpha)$ and $hR \in N[X]/(Q)$. Since h is invertible in $N_{\mathrm{supp}(\alpha)}[X]/(Q)$, we may assume that $R \in N[X]$. Then

$$(N_{\mathrm{supp}(\alpha)}[X]/(Q))_R = S_\alpha^{-1}(N[X]/(Q))_R,$$

where $S_\alpha = N \setminus \mathrm{supp}(\alpha)$ (cf. [6, p. 18]).
The algebra $(N_{\mathrm{supp}(\alpha)}[X]/(Q))_R$ is étale over $N_{\mathrm{supp}(\alpha)}$, and $(N[X]/(Q))_R$ is étale over N.

By construction, there exists a homomorphism $(N[X]/(Q))_R \to N$. Set t as the image of X under this homomorphism. In this case, the image of t by $N \longrightarrow N_{\mathrm{supp}(\alpha)} \longrightarrow k(\alpha)$ coincides with the image of X by

$$(N_{\mathrm{supp}(\alpha)}[X]/(Q))_R \longrightarrow S_\alpha^{-1}(N[X]/(Q))_R \longrightarrow k(\alpha).$$

Thus, this is x. Therefore the residue field of $N_{\mathrm{supp}(\alpha)}$ is $k(\alpha)$.

Now we will show that the ring $N_{\mathrm{supp}(\alpha)}$ is henselian. Let P be a polynomial with coefficients in $N_{\mathrm{supp}(\alpha)}$ and x be a simple root of $\bar{P}$ in $k(\alpha)$. Furthermore, let m_P be the kernel of the homomorphism

$$N_{\mathrm{supp}(\alpha)}[X]/(P) \ni X \mapsto x \in k(\alpha).$$

Take R as an element of B as mentioned above. In the same way, we may assume that $(N_{\mathrm{supp}(\alpha)}[X]/(P))_R = S_\alpha^{-1}((N[X]/(P))_R)$. Whence the image t of X under $(N[X]/(P))_R \longrightarrow N$ satisfies the equation

$$P(t) = 0$$

by the definition.
Therefore, $N_{\mathrm{supp}(\alpha)}$ is henselian.

(ii) We next prove an universal property. Let $f : N \longrightarrow B$ be a real closed local N-algebra with its maximal cone μ_B such that $f^{-1}(\mu_B) = \alpha$.

Then it is obvious that there exists only one N-algebra homomorphism $N_{\mathrm{supp}(\alpha)} \longrightarrow B$ such that this homomorphism and f commute.

(b): We will show (b). We need the following two lemmas to show (b).

Lemma 2.5. *Let N be a ring which satisfies an idempotency property, and let $\mathcal{A}$ be a sheaf given by*

$$\mathcal{A}(U) = \Big\{ s : U \longrightarrow \coprod_{\alpha \in U} \mathcal{N}_\alpha;\ \forall \alpha \in U,\ \exists a, f \in N,\ \forall \beta \in \mathcal{U}(f) \cap U,\ s(\beta) = a/f \ \text{ in } \ N_\beta \Big\}.$$

Then $\mathcal{A} \cong \mathcal{N}_{\mathrm{Spec}_r N}$.

Proof. We first construct a natural morphism $\mathcal{A} \longrightarrow \mathcal{N}_{\mathrm{Spec}_r N}$.

Let $\{(\mathcal{U}(\bar{f}_i) \cap U, \bar{a_i}/\bar{f}_i)\}$ be an element of $\mathcal{A}(U)$. By taking a sufficiently fine open covering of $\mathcal{U}(\bar{f}_i) \cap U$, $\mathcal{U}(\bar{f}_i) \cap U = \bigcup_{j \in J} V_j$, we may assume that there exists a_i which is an element of standard étale N-algebra B and a representation of $\bar{a_i}$, and f_i which is an element of standard étale N-algebra B and a representation of $\bar{f}_i$. Then a_i/f_i is an element of standard étale N-algebra B_{f_i}. Thus we obtain an element $\{(V_i, a_i/f_i, B_{f_i})\}$ in $\mathcal{N}_{\mathrm{Spec}_r N}(U)$.

It is obvious, by the definition of $\mathcal{N}_{\mathrm{Spec}_r N}(U)$, that any choice of representations of a_i/f_i gives the same element $\{(V_i, a_i/f_i, B_{f_i})\}$ of $\mathcal{N}_{\mathrm{Spec}_r N}(U)$ that another choice gives.

Every stalk of one sheaf already coincides with one of another sheaf, so we have proved this lemma. □

Lemma 2.6. *If $b_1, \ldots, b_k$ are elements of $\mathcal{N}(\mathcal{U}(f_1, \ldots, f_p))$, the element $1 + b_1^2 + \cdots + b_k^2$ is an invertible element in $\mathcal{N}(\mathcal{U}(f_1, \ldots, f_p))$.*

Proof. There exist an open covering $\{U_i\}_{i \in I}$ of $\mathcal{U}(f_1, \ldots, f_p)$, rings C_i which is étale over $N_{f_1,\ldots,f_p}$ and elements ${b'}_1, \ldots, {b'}_k$ of C_i such that $(U_i, {b'}_i, C_i)$ represents b_i on U_i. For any $\alpha \in \mathcal{U}(f_1, \ldots, f_p)$, $(1 + b_1^2 + \cdots + b_k^2)(\alpha) > 0$. Furthermore, the ring $C_{i(1+b_1^2+\cdots b_k^2)(\alpha)}$ is étale over N ([6, p. 16]). The element $1 + b_1^2 + \cdots b_k^2$ is invertible in the ring $C_{i(1+b_1^2+\cdots b_k^2)(\alpha)}$. Since $\mathcal{N}(\mathcal{U}(f_1, \ldots, f_p))$ satisfies an idempotency property,

$$(1 + b_1^2 + \cdots + b_k^2) \in \mathcal{N}(\mathcal{U}(f_1, \ldots, f_p))$$

is invertible by construction. □

Let

$$\psi : N_{f_1,\ldots,f_p} \longrightarrow \mathcal{N}(\mathcal{U}(f_1, \ldots, f_p))$$

be a natural homomorphism. In this situation, we will show that ψ is an isomorphism.

We will show that ψ is injective. We first show that the homomorphism ψ is injective in the case when $p = 1$.

Set $f := f_1$. Now assume that $\psi(a/f^n) = \psi(b/f^m)$. There exists an element h of $\mathrm{supp}(\alpha)$ such that $h(f^m a - f^n) = 0$ in N because the images of a/f^n and b/f^m coincide in N_α for any $\alpha \in \mathcal{U}(f)$.

Let $J \subset N$ be an annihilator of $f^m a - f^n b$.

Then $h \in J$, $h \notin \mathrm{supp}(\alpha)$. Therefore, $J \not\subset \mathrm{supp}(\alpha)$ for all $\alpha \in \mathcal{U}(f)$.

Since $Z(J) \hookrightarrow Z(f)$, $f \in {}^R\sqrt{J}$, where $Z(J)$ denotes the zero set of an ideal J. Now we conclude that

$$f_i^{2a_i} + b_1^2 + \cdots + b_k^2 \in J.$$

By Lemma 2.6, $f^a(f^m a - f^n b) = 0$ in $N(\mathcal{U}(f))$. Finally, we get the conclusion that

$$a/f^n = b/f^m.$$

Therefore ψ is injective.

The morphism

$$N_{f_1,f_2} \hookrightarrow (\mathcal{N}_{\mathrm{Spec}_r(\mathcal{N}(\mathcal{U}(f_1)))}(\mathcal{U}(f_2)))_{f_1} \hookrightarrow \mathcal{N}(\mathcal{U}(f_1, f_2))$$

is injective by Proposition 2.2. Therefore ψ is also injective in general case.

We will show the surjectivity of ψ. By idempotency,

$$(\mathcal{U}(f_1, \ldots, f_{p-1}), \mathcal{N}|_{\mathcal{U}(f_1,\ldots,f_{p-1})}) \cong (\mathrm{Spec}_r \mathcal{N}(\mathcal{U}(f_1, \ldots, f_{p-1})), \mathcal{N}_{\mathrm{Spec}_r \mathcal{N}(\mathcal{U}(f_1,\ldots,f_{p-1}))}).$$

Hence we have only to prove that the homomorphism $\psi : N_f \longrightarrow \mathcal{N}(\mathcal{U}(f))$ is surjective.

Choose an arbitrary $s \in \mathcal{N}(\mathcal{U}(f))$. The element s may be represented as $(a_i/h_i, \mathcal{U}(h_i))$ by Lemma 2.5.

In this situation, the equality

$$a_i/h_i = a_{i'}/h_{i'}$$

holds true on $\mathcal{U}(h_i, h_{i'})$. Since ψ is injective, $a_i/h_i = a_{i'}/h_{i'}$ in $N_{h_i,h_{i'}}$. Hence

$$(h_i h_{i'})^p (h_{i'} a_i - h_i a_{i'})(1 + b_1^2 + \cdots + b_k^2) = 0 \qquad (b_l \in N_{h_i,h_{i'}}).$$

Therefore

$$(h_i h_{i'})^p (h_{i'} a_i - h_i a_{i'}) = 0$$

by Lemma 2.6 and injectivity of ψ.

If p is an even number, exchange h_i^{p+1} for h_i and $h_i^p a_i$ for a_i, and if p is an odd number, exchange h_i^{p+2} for h_i and $h_i^{p+1} a_i$ for a_i. The same exchange of $h_{i'}$, $a_{i'}$ as before make us assume that s has a representation of the form a_i/h_i on $\mathcal{U}(h_i)$ such that $h_{i'} a_i = h_i a_{i'}$. By construction, $\mathcal{U}(f) \subset \cup\, \mathcal{U}(h_i)$. Therefore $Z(h_1, \ldots, h_k) \hookrightarrow Z(f)$, so $f \in {}^R\sqrt{\sum(h_{it})}$.

Now the equality

$$f^{2m} + b_1^2 + \cdots + b_k^2 = \sum c_i h_i \ (c_l, b_l \in N)$$

holds true. Set $a = \sum c_i a_i$. Then

$$\begin{aligned} h_{i'} a &= \sum c_i a_i h_{i'} = \sum c_i h_i a_{i'} \\ &= (f^{2m} + b_1^2 + \cdots b_k^2) a_j. \end{aligned}$$

Now we conclude that

$$a/(f^{2m} + b_1^2 + \cdots b_k^2) = a_j/h_j.$$

There exists $a' \in N$ such that $a'/f^n = a_j/h_j$ by Lemma 2.6.

Hence, $\psi(a'/f^n) = s$. In short, ψ is surjective. □

3. Separateness.

Definition 3.1. Let S be a real scheme, and X, Y be real schemes over S, i.e., real schemes with morphism to S. We define the real scheme $X \times_S Y$ with morphisms $p_1 : X \times_S Y \longrightarrow X$ and $p_2 : X \times_S Y \longrightarrow Y$ as follows:

For any real scheme Z such that the following diagram commutes,

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

there exists a morphism $\theta : Z \longrightarrow X \times_S Y$ such that $p_1 \circ \theta = f$, $p_2 \circ \theta = g$.

One can show in the same way as [3, Theorem 3.3] that, for any two real schemes X and Y over a real scheme S, $X \times_S Y$ exists uniquely up to isomorphism.

Definition 3.2. Let $f : X \longrightarrow Y$ be a morphism of real schemes.

The diagonal morphism is the unique morphism $\Delta : X \longrightarrow X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \longrightarrow X$ satisfies $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{id}$.

The morphism f is said to be separated if $\operatorname{im} \Delta$ is constructive and closed.

Furthermore, X is said to be separated if the morphism $f : X \longrightarrow \operatorname{Spec}_r \mathbf{Z}$ is separated.

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a morphism of real schemes. Then the following conditions are equivalent.*

- *The morphism f is separated.*
- *For any real closed field K, whose proper cone is β, and for any β-convex valuation ring R with quotient field K, we set $T = \operatorname{Spec}_r R$, $U = \operatorname{Spec}_r K$ and $i : U \longrightarrow T$ as a morphism induced by the inclusion $R \hookrightarrow K$.*

  Then for any morphism $q : T \to Y$ and any morphism $p : U \to X$ such that the following diagram commutes,

$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[q]{} & Y \end{array}$$

  there exists at most one morphism $r : T \longrightarrow X$ such that $p = r \circ i$.

Claim 1. *Under the condition of this theorem, T consists of two points and one is a specialization of another.*

Proof of Claim 1. By [4, Proposition 10.1.6, Proposition 10.2.3], T contains at least two points and the one is a specialization of another one.

Furthermore, one point is $\beta \cap R$ and the other is one induced by $B \longrightarrow B/m_B = k_B$. Now we assume that there exists more than two proper cones. By [4, Theorem 10.1.10], K has more than two proper cones. Contradiction.

Hence k_B is a real closed field, and the proof of Claim 1 is completed. □

Claim 2. *Under the condition of this theorem, to give $U \longrightarrow X$ is equivalent to giving a point x_1 of X and an inclusion $k(x_1) = \mathcal{N}_{x_1}/m_{x_1} \hookrightarrow K$. And to give $T \longrightarrow X$ is equivalent to giving points x_0, x_1 of X (where x_0 is a specialization of x_1), inclusions $k(x_1) = \mathcal{N}_{x_1}/m_{x_1} \hookrightarrow K$, $\mathcal{N}_{x_0} \hookrightarrow R$ and $m_{x_0} = M \cap \mathcal{N}_{x_0}$, provide that m_{x_0}, M is a maximal ideal of $\mathcal{N}_{x_0}, R$.*

Proof of Claim 2. The first part of this claim is obvious.

If the morphism $T \longrightarrow X$ is given, x_0, x_1 are given as the images of two points t_0, t_1 of T, where $t_0 \to t_1$.

The morphism $T \longrightarrow X$ can factorize through $Z = \{x_1\}^-$ as follows by Proposition 2.2:

$$T \longrightarrow Z = \{x_1\}^- \hookrightarrow X.$$

Therefore we get an inclusion which satisfies required conditions.

Assume that the inclusions are given. There exists a morphism $\mathrm{Spec}_r R \longrightarrow \mathrm{Spec}_r \mathcal{N} \longrightarrow X$ because there exists an inclusion $\mathcal{N}_{x_0} \longrightarrow R$. Hence we can easily construct required maps. The proof of Claim 2 is completed. □

We prove the theorem, using the above two claim.

Assume that f is separated and that there exist two morphism h, h' which satisfy the equations

$$h \circ i = p, \quad h' \circ i = p.$$

This two morphism construct a morphism

$$h'' : T \longrightarrow X \times_Y X.$$

Because h and h' coincide on U, $h''(t_1) \in \Delta(X)$. Furthermore, $t_0 \in \Delta(X)$ i.e., $h(t_0) = x_0 = h'(t_0)$, $h(t_1) = x_1 = h(t_1)$ because $\Delta(X)$ is closed and constructive. The inclusion $k(x_1) \subset K$ induces h, h'. Hence one of the statements is proved by Claim 2.

Conversely, assume that the condition of commutative diagram is satisfied. The set $\Delta(X)$ is constructive because a set is compact on $\mathrm{Spec}_r A$ if and only if it is constructive on $\mathrm{Spec}_r A$ [4, Corollary 7.1.13] and because an image of a compact set by a continuous map is compact. If $\Delta(X)$ is closed under specialization, $\Delta(X)$ is closed [4, Corollary 7.1.22]. Hence we have only to prove that $\Delta(X)$ is closed under specialization.

Choose an arbitrary element $\xi_1 \in \Delta(X)$, and let ξ_0 be a specialization of ξ_1. Set $K := k(\xi_1)$ and $\mathcal{N}_{\xi_0}$ as a stalk of the Nash structure sheaf at ξ_0 on $\{\xi\}^-$.

There exists a valuation ring R which dominates $\mathcal{N}_{\xi_0}$ by [5, p. 72 Theorem 10.1, 10.2]. We obtain a morphism

$$r : T = \mathrm{Spec}_r R \to X \times_Y X$$

such that $r(t_0) = \xi_0$ and $r(t_1) = \xi_1$ by Claim 2. Let p_1 and p_2 be the first and second projection of $X \times_Y X$ onto X, respectively. Then

$$f \circ p_1 \circ r = f \circ p_2 \circ r.$$

Therefore the morphism $T \longrightarrow X \times_Y X$ factorizes as

$$T \longrightarrow X \xrightarrow{\Delta} X \times_Y X.$$

Therefore, $\xi_0 \in \Delta(X)$.

The proof of this theorem is completed. □

Corollary 3.4. *In the same condition of Theorem* 3.3,

1. *Open and closed inclusions are separated.*
2. *If f and g are separated morphisms, then $f \circ g$ is separated.*
3. *If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are separated morphisms over S, then the morphism $f \times f' : X \times_S X' \longrightarrow Y \times_S Y'$ is separated.*
4. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms. If $g \circ s$ is separated, then f is separated.*
5. *A morphism $f : X \longrightarrow Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is separated for each i.*

Lemma 3.5. *Let X be a separated real scheme over an affine scheme S, and U, V be open affine subsets of X, then $U \cap V$ is affine.*

Proof. The composition map $X \xrightarrow{\Delta} X \times_S X \xrightarrow{p_1} X$ is an identity map.

Therefore $\Delta : X \longrightarrow \Delta(X)$ is a homeomorphism.

The set $\Delta(U \cap V) = \Delta(X) \cap (U \times_S V)$ is closed in the set $U \times_S V$. The real scheme $U \times_S V$ is affine by the way of construction. Therefore it is compact. Since Δ is a homeomorphism, we may assume that the set $U \cap V \subset U \times_S V$ is closed. Since $U \cap V$ is compact, it is constructive by [4, Corollary 7.1.13]. Hence $U \cap V$ is affine because it is a closed constructive subset of an affine real scheme. □

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