On a conjecture of W. Bergweiler

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Abstract: In this paper it is discussed for which meromorphic functions f the homogeneous differential $f(z)f''(z) - a(f'(z))^2$ has only finitely many zeros. It is shown that any transcendental meromorphic functions $f(z)$ have the form $R(z)\exp(P(z))$ for a rational function R and a polynomial P with the property if $a \neq 1$, $(n \pm 1)/n$, $n \in N$. This result settles one conjecture proposed by W. Bergweiler.

Key words: Differential polynomial; normal family; meromorphic function.

1. Introduction and main results. W. K. Hayman (see [1]) obtained in 1959 that if f is a transcendental entire function and $f(z)f''(z) \neq 0$, then f has the form $f(z) = \exp(\alpha z + \beta)$, where $\alpha\,(\neq 0)$ and $\beta \in C$. Afterwards E. Mues (see [2]) obtained the result in the case for meromorphic function of finite lower order and recently J. K. Langley (see [3]) proved it completely. If we put $f(z) = 1/(g(z))$, where $g(z)$ is an entire function, then $f''(z) = (2g\prime 2 - gg'')/(g^3)$ so that W. Hayman raised the open problem of whether differential polynomials such as $G(z) = g(z)g''(z) - 2(g'(z))^2$ in the transcendental entire function $g(z)$ necessarily have zeros except when $g(z) = \exp(az + b)$, (see [4]).

In 1978, E. Mues (see [5]) settled the open problem completely and showed that the problem does not hold if $a = 1$ by some examples like $f(z) = \cos z$. Allowing f to be meromorphic, W. Bergweiler in 1995 (see [6]) conjectured if f is a transcendental meromorphic function which is not of the form $f(z) = \exp(\alpha z + \beta)$ and if $a \neq 1$ and $a \neq (n+1)/n$, then $f(z)f''(z) - a(f'(z))^2$ has at least one zero. Moreover, he obtained the following result.

Theorem A. *Let f be a meromorphic function of finite order and let a be as above. If $f(z)f''(z) - a(f'(z))^2$ has only finitely many zeros, then f has the form $f(z) = R(z)\exp(P(z))$ for a rational function R and a polynomial P.*

Theorem B. *Let f be a transcendental meromorphic function of finite order and let a be as above. If $f(z)f''(z) - a(f'(z))^2$ has no zero, then f is of the form $f(z) = \exp(\alpha z + \beta)$, where $\alpha, \beta \in C$.*

Recently, J. K. Langley (see [7]) improved the above results in the case for function of finite lower order. In this paper, we will exclude the additional order restriction.

Theorem 1. *Let f be a meromorphic function in the complex plane and let $a \neq 1$, $(n \pm 1)/n$, where $n \in N$. If $f(z)f''(z) - a(f'(z))^2$ has only finitely many zeros, then f has the form $f(z) = R(z)\exp(P(z))$ for a rational function R and a polynomial P.*

Corollary. *Let f be a meromorphic function in the complex plane and let $a \neq 1$, $(n \pm 1)/n$, where $n \in N$. If $f(z)f''(z) - a(f'(z))^2$ has no zero, then f has one of the following forms*:

(i) $f(z) = \exp(\alpha z + \beta)$,

(ii) $f(z) = \alpha z + \beta$,

(iii) $f(z) = \frac{1}{(\alpha z+\beta)^n}$,

where $\alpha\,(\neq 0)$, $\beta \in C$.

2. A main proposition and some lemmas. For a function f meromorphic in a domain D we shall use the notation

$$M_f := f'(f^{-1}(0)) = \{f'(z) : z \in D \text{ and } f(z) = 0\}.$$

For $r > 0$ and $a \in C$ we put $D(a, r) := \{z \in C : |z - a| < r\}$.

In 1980, Y. X. Gu (see [8]) proved that the family of meromorphic functions in one domain D is normal if for each $f \in F$, $f(z) \neq 0$ and $f'(z) \neq 1$ in D. Recently, W. Bergweiler (see [9]) extended the above result by allowing f to have zeros. In this paper, we obtain more general result as follows:

Proposition. *Let A, B and ε be positive real numbers. Let F be the family of all functions f mero-*

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morphic in D which satisfy the following conditions:

1) *If* $f'(z) = 1$, *then* $|f(z)| \geq A$,
2) *If* $f(z) = 0$, *then* $0 < |f'(z)| \leq B$,
3) *If* Δ *is a disk in* D *and if* f *has* $m \geq 2$ *zeros* $z_1, z_2, \ldots, z_m \in \Delta$, *then*

$$\left|\sum_{j=1}^{m} f'(z_j)^{-1} - 1\right| \geq \varepsilon. \tag{1}$$

Then F *is normal in* D.

To prove the proposition, we need some lemmas.

Lemma 1 ([10]). *Let* $g(z)$ *be a transcendental meromorphic function with finite order. If* $g(z)$ *has only finitely many critical values, then* $g(z)$ *has only finitely many asymptotic values.*

Lemma 2 ([11]). *Let* $g(z)$ *be a transcendental meromorphic function and suppose that the set of all finite critical and asymptotic values of* $g(z)$ *is bounded. Then there exists* $R > 0$ *such that if* $|z| > R$ *and* $|g(z)| > R$, *then*

$$|g'(z)| \geq \frac{|g(z)|}{16\pi|z|} \log|g(z)|.$$

Lemma 3 ([12]). *Let* $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + (p(z))/(q(z))$, *where* $a_0, a_1, \ldots, a_n$ *are constants with* $a_n \neq 0$, $p(z)$ *and* $q(z)$ *are two coprime polynomials with* $\deg p(z) < \deg q(z)$, k *be a positive integer. If* $f^{(k)}(z) \neq 1$, *then*

$$f(z) = \frac{1}{k!} z^k + \cdots + a_0 + \frac{1}{(az+b)^m},$$

where $a\,(\neq 0)$, b *are constants,* m *is a positive integer.*

Lemma 4 ([13]). *Let* F *be a family of meromorphic functions on the unit disc* Δ, *all of whose zeroes have multiplicity at least* k, *and suppose there exists* $A \geq 1$ *such that* $|f^{(k)}(z)| \leq A$ *whenever* $f(z) = 0$, $f \in F$. *Then if* F *is not normal, there exist, for each* $0 \leq \alpha \leq k$,

a) *a number* r, $0 < r < 1$,
b) *points* z_n, $|z_n| < r$,
c) *functions* $f_n \in F$, *and*
d) *positive numbers* $\rho_n \to 0$

such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g *is a meromorphic function on* C *such that* $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

Lemma 5. *Let* f *be a transcendental meromorphic function with finite order, all of whose zeros are of multiplicity* (*at least*) k, *and let* A *be a positive real number. If* $|f^{(k)}(z)| \leq A$ *when* $f(z) = 0$, *then for each* l, $1 \leq l \leq k$, $f^{(l)}(z)$ *assumes any finite nonzero value infinitely often.*

Proof. Suppose that $f^{(l)}(z)$ assumes nonzero value b finitely many.

Define $g(z) = f^{(l-1)}(z) - bz$, then $g'(z) = f^{(l)}(z) - b$ only has finitely many zeros. By Hayman's inequality (see [4, Theorem 3.5]), we see that $f(z)$ has infinitely many zeroes, $z_1, z_2, \ldots$, and $\lim_{n\to\infty} z_n = \infty$.

Since all zeros of $f(z)$ are of multiplicity (at least) k, and $|f^{(k)}(z)| \leq A$ when $f(z) = 0$, we have

$$g(z_n) = -bz_n, \quad |g'(z_n)| = |f^{(l)}(z_n) - b| \leq M, \tag{2}$$

where $M = |b|$ when $l < k$ or $M = A + |b|$ when $l = k$.

On the other hand, since $g'(z) = f^{(l)}(z) - b$ has only finitely many zeros, by Lemma 1 we know that $g(z)$ has only finitely many asymptotic values. Thus, by Lemma 2 we deduce that

$$|g'(z_n)| \geq \frac{|g(z_n)|}{16\pi|z_n|} \log|g(z_n)|, \tag{3}$$

we get by (2) and (3) that

$$|M| \geq \frac{|b|}{16\pi} \log|bz_n|. \tag{4}$$

On the right hand of (4), we find that $\log|bz_n| \to \infty$ as $n \to \infty$. This is a contradiction. Therefore the conclusion of the lemma holds. □

Remark. The lemma does not hold for $l = 0$, where $f^{(0)} :\equiv f$ when $l = 0$. For instance, let $f(z) = (A(1 + Be^{-2z}))/(1 - Be^{-2z})$, where A and B are nonzero constants. Then $f'(z) = A$ when $f(z) = 0$, but $f(z) \neq \pm A$.

Proof of Proposition. Suppose that F is not normal in D. Lemma 4 implies that there exist $\{f_n\} \subset F$, $z_n \to z_0 (z_0 \in D)$, $\rho_n \to 0$ such that

$$g_n(\xi) := \frac{f_n(z_n + \rho_n \xi)}{\rho_n} \to g(\xi) \tag{5}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function. Moreover, $g^{\#}(\xi) \leq g^{\#}(0) = B + 1$

By (5), we have

$$g_n'(\xi) = f_n'(z_n + \rho_n \xi) \to g'(\xi). \tag{6}$$

Suppose that $g(\xi)$ is a polynomial. We distinguish two cases.

Case 1. Suppose that $\deg g \geq 2$. Let $g'(w_0) = 1$, by Hurwitz's Theorem, there exists a sequence $\xi_n, \xi_n \to w_0$, and

$$g'_n(\xi_n) = f'_n(z_n + \rho_n \xi_n) = 1.$$

Note that $|f(z)| \geq A$ when $f'(z) = 1$, by (5) we obtain $g(w_0) = \infty$. This is a contradiction.

Case 2. Suppose that $\deg g = 1$, i.e., $g(\xi) = az + b$. It follows that $|a| \leq B$ and hence that

$$g^{\#}(0) = \frac{|a|}{1 + |b|^2} \leq |a| < B + 1,$$

which is also contradiction. This implies that $g(\xi)$ is not a polynomial. Moreover, we obtain that $g'(\xi) \neq 1$ as above proof.

Let $g(\xi_0) = 0$. Again using Huriwitz's Theorem and (6), from Condition 2) we can similarly derive that $M_g \subset D(0, B)$. Since $g(\xi)$ is not a polynomial, Lemma 5 implies that

$$g(\xi) = r(\xi) + \frac{p(\xi)}{q(\xi)},$$

where $r(\xi), p(\xi), q(\xi)$ are polynomials, $p(z)$ and $q(z)$ are coprime polynomials with $\deg q > \deg p$. Therefore, by Lemma 3 we have

$$(7) \qquad g(\xi) = \xi + a + \frac{b}{(\xi + c)^l},$$

where $a, b, c \in C$, $b \neq 0$, $l \in N$.

Set $m := l + 1$ and $R > \max_{1 \leq j \leq m} |\xi_j|$, where for $1 \leq j \leq m$, ξ_j is the zero of $g(\xi)$, counted according to multiplicity. For large n we have m distinct zeros $\xi_{j,n} \in D(0, R)$ of $g_n(\xi)$ such that $\xi_{j,n} \to \xi_j$ for $1 \leq j \leq m$. Write $\zeta_{j,n} := z_n + \rho_n \xi_{j,n}$, then $\zeta_{j,n}$ $(1 \leq j \leq m)$ are the zeros of f_n. Moreover, $\zeta_{j,n} \in \Delta_n := D(z_n, \rho_n R)$ for $1 \leq j \leq m$, and for sufficiently large n, $\Delta_n \subset D$ and f_n has no further zeros in Δ_n. Therefore, by (6) we have

$$(8) \quad \sum_{j=1}^{m} f'_n(\zeta_{j,n})^{-1} = \sum_{j=1}^{m} g'_n(\xi_{j,n})^{-1} = \sum_{j=1}^{m} res\left(\frac{1}{g_n}, \xi_{j,n}\right) \to \sum_{\xi \in g^{-1}(0)} res\left(\frac{1}{g}, \xi\right).$$

On the other hand, we have from (7) that

$$\frac{1}{g(\xi)} = \frac{1}{\xi} + O\left(\frac{1}{\xi^2}\right)$$

as $\xi \to \infty$, and hence, by (7), (8) we obtain

$$\sum_{j=1}^{m} f'_n(\zeta_{j,n})^{-1} \to 1$$

as $n \to \infty$. This is contradiction. Therefore, the conclusion of Proposition holds. □

3. Proof of Theorem 1. We start with the following definition and lemma.

Definition 1. A meromorphic function f on C is called a normal function if there exists a positive number M such that

$$f^{\#}(z) \leq M$$

Here, as usual, $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ denotes the spherical derivative.

From Definition 1, we obtain

Lemma 6. *A normal meromorphic function has order at most* 2. *Especially, a normal entire function has order at most* 1.

For proof of theorem, we also need the following lemmas.

Lemma 7. *Let $\{a_m\}$ be an integer sequence and $a \neq 1 \pm (1/n)$, where $n \in N$. Then there exists a positive number ε such that for each a_m,*

$$|a_m(a - 1) - 1| \geq \varepsilon.$$

In fact, since $a \neq 1 \pm (1/n)$, we deduce that $a_m(a - 1) - 1 \neq 0$ for each a_m. Next, denote by $\{a_n^k\}$ the sequence of all $a_m(a - 1) - 1$ by increasing modulus, i.e., $0 < |a_m^1| \leq |a_m^2| \leq \cdots \leq |a_m^k| \leq \cdots$. Obviously, the conclusion of lemma holds as that we choose $\varepsilon = |a_m^1|$.

Lemma 8. *Let f be a meromorphic function in the plane C and let $a \neq 1$, $(n \pm 1)/n$, where $n \in N$. If $f(z)f''(z) - a(f'(z))^2$ has only finitely many zeros, then $h(z)$ is a normal function, where*

$$h(z) := \frac{f(z)}{(1 - a)f'(z)}.$$

Proof. Suppose that $h(z)$ is not a normal function in C. Then there exists a sequence z_n such that $h^{\#}(z_n) \to \infty$. Define

$$(9) \quad h_n(z) := h(z + z_n) = \frac{f(z + z_n)}{(1 - a)f'(z + z_n)}, \quad n \in N.$$

Then we obtain that $\{h_n\}_1^{\infty}$ is not normal at $z_0 = 0$.

Now we shall discuss the normality of the family $\{h_n\}_1^{\infty}$ in Δ as follows:

From (9), we have

(10)

$$h'_n(z) = \frac{1}{1-a}\,\frac{f'(z+z_n)^2 - f(z+z_n)f''(z+z_n)}{f'(z+z_n)^2} = \frac{af'(z+z_n)^2 - f(z+z_n)f''(z+z_n)}{(1-a)f'(z+z_n)^2} + 1.$$

Let ξ be a zero of $h_n(z)$, then ξ is either a zero or a pole of $f(z+z_n)$. If ξ is a zero of $f(z+z_n)$ with multiplicity l, we get that $h'_n(\xi) = 1/(l(1-a))$. If ξ is a pole of $f(z)$ with multiplicity m, we have $h'_n(\xi) = 1/(m(a-1))$. Hence, $0 < |h'_n(z)| \le 1/(|a-1|)$ when $h_n(z) = 0$.

On the other hand, since $af'(z)^2 - f(z)f''(z)$ has only finitely many zeros, we denote them by $\{w_1, w_2, \ldots, w_k\}$ and define $A := \min_{1\le i\le k}\{|h(w_i)|\}$. Noting that $a \ne 1$, $(n \pm 1)/n$ and $h'(w_i) = 1$, we can deduce that $h(w_i) \ne 0$ for $i = 1, 2, \ldots, k$, and hence $A > 0$. Thus, we have from (10) that $|h_n(z)| \ge A$ when $h'_n(z) = 1$ for $n \in N$.

Moreover, suppose that $\Delta_1 \subset \Delta$ is a disk and h_n has m zeros $u_1, u_2, \ldots, u_m$ in Δ_1. As above, we obtain

$$\left|\sum_{j=1}^{m} h'_n(u_j)^{-1} - 1\right| = |a_m(a-1) - 1|, \tag{11}$$

where a_m is an integer number.

By Lemma 7, there exists a positive number ε such that for h_n,

$$\left|\sum_{j=1}^{m} h'_n(u_j)^{-1} - 1\right| \ge \varepsilon.$$

Above all, by Proposition, we deduce that $\{h_n\}_1^\infty$ is normal in Δ. This is a contradiction to that $\{h_n\}_1^\infty$ is not normal at 0. Therefore, the conclusion of Lemma 8 holds. □

Now we are going to prove Theorem 1.

Define $h := f/((1-a)f')$. Lemma 8 implies that $h(z)$ is a normal function in C. Therefore, $\lambda(h) \le 2$.

Similar to the proof of Lemma 8, we get that $h(z)$ satisfies the condition of Lemma 5. Therefore, $h(z)$ is a rational function, i.e., $1/(h(z))$ is also a rational function with no multiple pole points. Therefore, we can deduce that $f(z)$ has the form $f(z) = R(z)\exp(P(z))$ for a rational function R and a polynomial P.

To prove the corollary, we have that $f(z)f''(z) - a(f'(z))^2 \ne 0$, proceed as above, we find that $h(z)$ is a rational function.

According to the assumption of corollary, we get that $h'(z) \ne 1$, and hence, by Lemma 3 we obtain that $h(z)$ has the form $h(z) = \alpha z + b$ or

$$h(z) = z + \beta + \frac{b}{(z+c)^l}, \tag{12}$$

where $\alpha\,(\ne 1)$, β, b, c are constants.

Suppose that $h(z) = z + \beta + b/((z+c)^l)$. Let $z_1, z_2, \ldots, z_m$ are the zeros of $h(z)$, here $m = l + 1$. Then as the proof of Proposition we obtain $\sum_{j=1}^m h'(z_j)^{-1} = 1$, but from the proof of Lemma 8, we know that there exists a positive number ε such that $|\sum_{j=1}^m h'(z)^{-1} - 1| \ge \varepsilon$. This gives a contradiction, and hence, $h(z) = \alpha z + b$.

Now we distinguish the following cases.

If $\alpha = 0$, then $f(z)$ has the form (i).

If $\alpha \ne 0$, by the definition of $h(z)$ we deduce that $f(z) = (\alpha z + \beta)^n$. Note that $f(z)f''(z) - a(f'(z))^2 \ne 0$, we obtain that $f(z)$ has the form (ii) or (iii).

This completes the proof of the corollary.

4. Remarks. As we know, one interesting problem on uniqueness of meromorphic function is what can be said about the case when a single finite value is shared by a meromorphic function and its first two derivatives. In 1986, G. Jank, E. Mues and L. Volkmann (see [15]) obtained.

Theorem C. *Let f be meromorphic in C, not a constant, and $a \in C \setminus \{0\}$. If f, f', f'' share the value a CM (counting multiplicities), then $f = f'$.*

Afterwards, K. Tohge discussed the case when $a = 0$ by adding some conditions (see [16]).

It follows from the proof of the corollary and Lemma 8 that if we define $h := (f/f')$, then we can deduce that h satisfies the condition of Proposition when f, f', f'' share 0 IM (ignoring multiplicities). We thus also obtain the following result.

Theorem 2. *Let f be meromorphic in C, not a constant. If f, f', f'' share the value 0 IM (ignoring multiplicities), then $f = \exp(Az + B)$ or $(Az+B)^n$, where $n \in Z \setminus \{0, 1, 2\}$ and $A\,(\ne 0)$, $B \in C$.*

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