A note on regularity of Noetherian complete local rings of unequal characteristic

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Abstract: Let $(R, \boldsymbol{m})$ be a Noetherian complete local ring with unequal characteristic, and let (P, pP) be a discrete valuation ring contained in R. Then, under some assumptions of separability on the residue fields, the following conditions are equivalent: (1) R is a regular local ring and $p \notin \boldsymbol{m}^2$. (2) The $\boldsymbol{m}$-adic higher differential algebra $\widehat{D}_t(R/P, \boldsymbol{m})$ is a polynomial ring over R for some t $(1 \leq t)$.

Key words: Regular local ring; $\boldsymbol{m}$-adic higher differential algebra.

1. Introduction. The note is a continuation of [FN2]. In [NS], Y. Nakai and S. Suzuki showed the following theorem:

Let $(R, \boldsymbol{m}, K)$ be a Noetherian complete local ring with $\mathrm{char}(R) = 0$ and $\mathrm{char}(K) = p \neq 0$, and let (P, pP, k) be a discrete valuation ring contained in R. Then, under some assumptions of separability on the residue fields K and k, the following conditions are equivalent:

(1) R is a regular local ring and $p \notin \boldsymbol{m}^2$.

(2) The module $\widehat{D}_P(R)$ of $\boldsymbol{m}$-adic P-differentials in R is a free R-module.

In the paper [FN2], we showed that these conditions (1) and (2) are equivalent with

(3) The $\boldsymbol{m}$-adic higher differential algebra $\widehat{D}_t(R/P, \boldsymbol{m})$ of R/P of length t is a polynomial ring over R for every t $(t = 1, 2, \ldots)$.

The purpose of this note is to prove that the condition (3) is also equivalent with

(4) The algebra $\widehat{D}_t(R/P, \boldsymbol{m})$ is a polynomial ring over R for some t $(1 \leq t)$.

2. Preliminaries. All rings in this paper are commutative rings with identity elements. A ring homomorphism will always a ring homomorphism which sends identity element to identity element. We always denote by t a natural number.

Let P be a ring, R a P-algebra with a ring homomorphism $\rho : P \to R$ and $\boldsymbol{m}$ an ideal of R.

Let S be an R-algebra with a ring homomorphism $f : R \to S$. For an integer $n \geq 1$, by a higher P-derivation of length n from R into S, we mean a sequence $(D_0, D_1, \ldots, D_n)$ of mappings $D_i : R \to S$ such that

(1) $D_0 = f$,

(2) $D_i(ab) = \sum_{j=0}^{i} D_j(a)D_{i-j}(b)$, $D_i(a + b) = D_i(a) + D_i(b)$ for any $a, b \in R$ and $i \geq 0$,

(3) $D_i\rho = 0$ for all $i \geq 1$.

We denote the set of all higher P-derivations of length n from R into S by $H\,\mathrm{Der}_P^n(R, S)$.

We shall say that R is an $\boldsymbol{m}$-adic ring or R has the $\boldsymbol{m}$-adic topology, if R has the topology with the fundamental system of neighborfoods of zero $\{\boldsymbol{m}^r \mid r = 1, 2, \ldots\}$. Let A be an R-algebra or an R-module. We shall say that A is an $\boldsymbol{m}$-adic R-algebra or an $\boldsymbol{m}$-adic R-module (or A has the $\boldsymbol{m}$-adic topology), if A has the topology with the fundamental system of neighborfoods of zero $\{\boldsymbol{m}^r A \mid r = 1, 2, \ldots\}$. The $\boldsymbol{m}$-adic topology of A is not necessarily Hausdorff. The $\boldsymbol{m}$-adic topology of A is Hausdorff if and only if $\bigcap_{r=0}^{\infty} \boldsymbol{m}^r A = (0)$.

We denote by $\widehat{D}_t(R/P, \boldsymbol{m})$ an $\boldsymbol{m}$-adic higher differential algebra of R over P of length t, that is, an R-algebra charaterized by the following conditions:

(1) $\widehat{D}_t(R/P, \boldsymbol{m})$ is a Hausdorff $\boldsymbol{m}$-adic R-algebra.

(2) There exists an element $\widehat{\boldsymbol{d}}_{R/P} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t) \in H\,\mathrm{Der}_P^t(R, \widehat{D}_t(R/P, \boldsymbol{m}))$ ($\widehat{\boldsymbol{d}}_{R/P}$ is called the associated derivation of $\widehat{D}_t(R/P, \boldsymbol{m})$).

(3) $\widehat{D}_t(R/P, \boldsymbol{m})$ is an R-algebra generated by $\{\hat{d}_n(a) \mid a \in R,\ n = 0, 1, \ldots, t\}$.

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(4) Let A be an arbitrary Hausdorff $\boldsymbol{m}$-adic R-algebra and $(D_0, D_1, \ldots, D_t) \in H\,\mathrm{Der}^t_P(R, A)$. Then there exists a ring homomorphism $g : \widehat{D}_t(R/P, \boldsymbol{m}) \to A$ such that $D_n = g\hat{d}_n$ for all n.

It is known that an $\boldsymbol{m}$-adic higher differential algebra of R/P of length t exists and uniquely determined up to isomorphism and up to homeomorphism. The R-algebra $\widehat{D}_t(R/P, \boldsymbol{m})$ is a graded R-algebra, $\widehat{D}_t(R/P, \boldsymbol{m}) = \bigoplus_{n=0}^{\infty} \widehat{D}_t(R/P, \boldsymbol{m})_n$, where $\widehat{D}_t(R/P, \boldsymbol{m})_n$ is the R-submodule of $\widehat{D}_t(R/P, \boldsymbol{m})$ generated by the homogeneous elements

$$\{\hat{d}_{n_1}(a_1) \cdots \hat{d}_{n_s}(a_s) \mid a_i \in R,\ 0 \le n_i \le t,\ n_1 + \cdots + n_s = n \ \text{ for some } \ s \ge 1\}.$$

The module $(\widehat{D}_P(R), d)$ of $\boldsymbol{m}$-adic P-differentials in R, defined in [NS], coincides with $(\widehat{D}_t(R/P, \boldsymbol{m})_1, \hat{d}_1)$ for every t.

3. Main result. The following lemma is a key to proof of our main result.

Lemma. *Let P be a ring, R a P-algebra and $\boldsymbol{m}$ an ideal of R. Put $K := R/\boldsymbol{m}$. We denote by Z the ideal $\boldsymbol{m} \bigoplus \bigoplus_{i=1}^{\infty} \widehat{D}_t(R/P, \boldsymbol{m})_i$ of the graded R-algebra $\widehat{D}_t(R/P, \boldsymbol{m})$. Then $(Z^2)_i := Z^2 \bigcap \widehat{D}_t(R/P, \boldsymbol{m})_i$ is an R-submodule of $\widehat{D}_t(R/P, \boldsymbol{m})_i$. Let A_i $(i = 2, \ldots, t)$ be the R-submodule of $\widehat{D}_t(R/P, \boldsymbol{m})_i$ generated by the set $\{xy \mid x \in \widehat{D}_t(R/P, \boldsymbol{m})_j, y \in \widehat{D}_t(R/P, \boldsymbol{m})_{i-j}, j = 1, \ldots, i-1\}$ and $A_1 := (0)$. Put $\hat{\boldsymbol{d}}_{R/P} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t)$. Then the mapping $\delta_i : R \to \widehat{D}_t(R/P, \boldsymbol{m})/\overline{A}_i$ $(x \mapsto \hat{d}_i(x) + \overline{A}_i)$ is a P-derivation of R into the R-module $\widehat{D}_t(R/P, \boldsymbol{m})_i/\overline{A}_i$ for every i $(i = 1, \ldots, t)$, where $\overline{A}_i$ is the closure of A_i in $\widehat{D}_t(R/P, \boldsymbol{m})_i$ with respect to the $\boldsymbol{m}$-adic topology. Furthermore we get the following:*

(1) *$(\widehat{D}_t(R/P, \boldsymbol{m})_i/\overline{A}_i, \delta_i) = (\widehat{D}_P(R), d)$ for every i $(i = 1, \ldots, t)$.*

(2) *There exists a K-module isomorphism*

$$\widehat{D}_t(R/P, \boldsymbol{m})_i/(Z^2)_i \simeq \widehat{D}_t(R/P, \boldsymbol{m})_1/\boldsymbol{m}\widehat{D}_t(R/P, \boldsymbol{m})_1$$

for every i $(i = 1, \ldots, t)$.

Proof. The proof is similar to that of [FN1, 2.5]. □

Now we are ready to prove our main result.

Theorem. *Let $(R, \boldsymbol{m}, K)$ be a complete Noetherian local ring with* $\mathrm{char}(R) = 0$ *and* $\mathrm{char}(K) = p \ne 0$, *and let (P, pP, k) be a discrete valuation ring contained in R. Assume that K is separably generated over k and* $\mathrm{Tr.deg}(K/k)$ *is finite. Then the following conditions are equivalent:*

(1) *R is a regular local ring and $p \notin \boldsymbol{m}^2$.*

(2) *$\widehat{D}_t(R/P, \boldsymbol{m})$ is a polynomial ring over R for every t $(t = 1, 2, \ldots)$.*

(3) *$\widehat{D}_t(R/P, \boldsymbol{m})$ is a polynomial ring over R for some t $(1 \le t)$.*

(4) *$\widehat{D}_P(R)$ is a free R-module.*

Proof. By Theorem 3.4 of [FN2], it only remains to show that (3) implies (4). Suppose that $\widehat{D}_t(R/P, \boldsymbol{m})$ is a polynomial ring over R for some t. From [FN2, 2.8], there are finite elements $\{a_1, \ldots, a_n\}$ of R with $n := \dim_K\big(\widehat{D}_t(R/P, \boldsymbol{m})_1/\boldsymbol{m}\widehat{D}_t(R/P, \boldsymbol{m})_1\big)$ such that $\widehat{D}_t(R/P, \boldsymbol{m})$ is generated by the tn elements $\{\hat{d}_i(a_1), \ldots, \hat{d}_i(a_n) \mid i = 1, \ldots, t\}$ as an R-algebra, where $\hat{\boldsymbol{d}}_{R/P} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_t)$. We denote by Z the ideal $\boldsymbol{m} \bigoplus \bigoplus_{i=1}^{\infty} \widehat{D}_t(R/P, \boldsymbol{m})_i$ of the graded R-algebra $\widehat{D}_t(R/P, \boldsymbol{m})$. Then $Z/Z^2 = \boldsymbol{m}/\boldsymbol{m}^2 \bigoplus \bigoplus_{i=1}^{t} Z_i/(Z^2)_i$, where $Z_i := \widehat{D}_t(R/P, \boldsymbol{m})_i$ and $(Z^2)_i$ are the same notations as in Lemma. Furthermore we have that $\dim_K Z/Z^2 = \dim_K \boldsymbol{m}/\boldsymbol{m}^2 + tn$ by Lemma. On the other hand, since $\widehat{D}_t(R/P, \boldsymbol{m})$ is a finitely generated R-algebra, there are finite variables $\{X_1, \ldots, X_s\}$ such that $\widehat{D}_t(R/P, \boldsymbol{m}) = R[X_1, \ldots, X_s] := R[X]$. We may assume that $X_j \in Z$ $(j = 1, \ldots, s)$. It follows that $Z = \boldsymbol{m}R[X] + (X_1, \ldots, X_s)$, and $\dim_K Z/Z^2 = \dim_K \boldsymbol{m}/\boldsymbol{m}^2 + s$ (cf. [O, 3.1]). Therefore $s = tn$. This means that $\widehat{D}_t(R/P, \boldsymbol{m})$ is the polynomial ring over R with variables $\{\hat{d}_i(a_1), \ldots, \hat{d}_i(a_n) \mid i = 1, \ldots, t\}$ by [ZS, Ch.I, Theorem 15]. Therefore $\widehat{D}_t(R/P, \boldsymbol{m})_1 = \widehat{D}_P(R)$ is the free R-module with a free basis $\{\hat{d}_1(a_1), \ldots, \hat{d}_1(a_n)\}$. □

We end this note by the following remarks.

Remarks. 1) In case of equal characteristic, an analogous result of the equivalency of (3) and (1) in Theorem is not true in general as follows:

Let k be a field of $\mathrm{char}(k) = p \ne 0$ and $R = k[X]/(X^p) := k[x]$. It is clear that R is a Noetherian complete local ring with the maximal ideal $\boldsymbol{m} = (x)$. Then $\widehat{D}_1(R/k, \boldsymbol{m})$ is a polynomial ring over R, $\widehat{D}_p(R/k, \boldsymbol{m})$ is not a polynomial ring over R and R is not a regular local ring.

2) As a corollary to Theorem 3.1 (resp. Theorem 3.4) of [FN2], we have another version than Corollary 3.2 (resp. Corollary 3.5) of [FN2]. Under the same notations as in [FN2], we have the following:

In Theorem 3.1 (resp. Theorem 3.4) of [FN2], let us remove the condition that R is complete, instead, let us assume that the differential module $\Omega_{R/k}$ of R/k (resp. the differential module $\Omega_{R/P}$ of R/P) is finitely generated. In this case, we have the same conclusion as in Theorem 3.1 (resp. Theorem 3.4) of [FN2]. Because from Section 2.5 of [FN1] and Section 2.4 of [FN2], the conditions of Corollary 3.2 (resp. Corollary 3.5) of [FN2] are satisfied.

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