# Invariants of two dimensional projectively Anosov diffeomorphisms

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**Abstract:** We define invariants of two dimensional projectively Anosov diffeomorphisms. More precisely, we show the space of circles tangent to the invariant subbundle has a kind of Morse decomposition and the homotopy type of its one point compactification is preserved under any homotopy of projectively Anosov diffeomorphisms. We also calculate the invariants for some examples.

**Key words:** Projectively Anosov systems; dominated splittings.

**1. Introduction.** Let $M$ be a manifold and $\|\cdot\|$ a norm on the tangent bundle $TM$. For a diffeomorphism $f$ on $M$ and its invariant set $\Lambda$, we call a continuous splitting $TM|_\Lambda = E^u \oplus E^s$ *a dominated splitting* associated to $f$ on $\Lambda$ if $Df$ preserves both $E^u$ and $E^s$ and there exist two constants $C > 0$ and $\lambda \in (0,1)$ such that

$$\|Df^n|_{E^s(z)}\| \cdot \|(Df^n|_{E^u(z)})^{-1}\| < C\lambda^n$$

for all $z \in \Lambda$ and $n \geq 1$. A non-trivial dominated splitting $TM = E^u_f \oplus E^s_f$ on the whole manifold is called a *projectively Anosov splitting* (or a simply $\mathbf{P}$A splitting) associated to $f$. We say a diffeomorphism $f$ is *projectively Anosov* (or simply $\mathbf{P}$A) when $f$ possesses a $\mathbf{P}$A splitting. Let $\mathbf{P}\mathrm{A}^r(M)$ be the set of all $\mathbf{P}$A diffeomorphisms on $M$. It is known that the $\mathbf{P}$A splitting $E^u_f \oplus E^s_f$ is unique for every two dimensional $\mathbf{P}$A diffeomorphism $f$.

A continuous family $\{f_t\}_{t\in[0,1]}$ of $C^1$ diffeomorphisms is called a $\mathbf{P}$A homotopy if all $f_t$ is contained in $\mathbf{P}\mathrm{A}^1(M)$. The main aim of this paper is to define invariants of two dimensional $\mathbf{P}$A diffeomorphisms which are preserved under $\mathbf{P}$A homotopies and to investigate their properties. Since the two dimensional torus $\mathbf{T}^2$ is the only orientable surface which admits $\mathbf{P}$A diffeomorphisms, we focus on $\mathbf{P}$A diffeomorphisms on $\mathbf{T}^2$.

Originally, Mitsumatsu [4] and Eliashberg and Thurston [2] have introduced the concept of $\mathbf{P}$A (or conformally Anosov) systems for three dimensional flows in order to investigate contact structures. They have given a natural correspondence between three dimensional $\mathbf{P}$A flows and bi-contact structures, which are pairs of mutually transverse positive and negative contact structures. Their correspondence induces a one-to-one correspondence between homotopy classes of them. In this view point, the study of $\mathbf{P}$A homotopy invariants of two dimensional $\mathbf{P}$A diffeomorphisms is a first step to classify three dimensional $\mathbf{P}$A flows and bi-contact structures.

The simplest invariant of $\mathbf{P}$A diffeomorphisms is the homotopy class as a continuous map. Since $E^u_f$ is continuous with respect to $f$, the homotopy class of $E^u_f$ as a line field on $\mathbf{T}^2$ is also an invariant. One of the natural problem is to find other non-trivial invariants. As we see later, the invariants defined here distinguish two $\mathbf{P}$A diffeomorphisms of which subbundles $E^u$ are contained in a same homotopy class of line fields.

To state our results, we introduce some definitions. We refer [3] for the standard terms on dynamical systems. A diffeomorphism $f$ is called *non-degenerate* if all periodic points are hyperbolic in the sense of dynamical systems. Note that non-degenerate diffeomorphisms are generic in the space of $C^r$ $\mathbf{P}$A diffeomorphisms by Kupka-Smale's theorem. We say an integer homology class $a \in H_1(\mathbf{T}^2, \mathbf{Z})$ is *prime* when $a \neq na'$ for all $n \geq 2$ and $a' \in H_1(\mathbf{T}^2, \mathbf{Z})$. For every topological space $X$, let $X \cup \{\infty\}$ denote the one point compactification of $X$. We say $\mathbf{P}$A diffeomorphism $f$ is *orientable* when both subbundles of the $\mathbf{P}$A splitting are orientable and $Df$ preserves orientations of them.

Let $\widetilde{\mathcal{C}}(E^u)$ denote the set of all $C^1$ maps $\gamma$

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from the circle $S^1$ to $\mathbf{T}^2$ which satisfy that $\dot{\gamma}(t) \in E^u(\gamma(t)) \setminus \{0\}$ for all $t \in S^1$. The group $\mathrm{Diff}^1_+(S^1)$ of the orientation preserving diffeomorphism on the circle acts on $\widetilde{\mathcal{C}}(E^u)$ by the composition from right. Let $\mathcal{C}(E^u)$ denote the quotient space $\widetilde{\mathcal{C}}(E^u)/\mathrm{Diff}^1_+(S^1)$ and $\pi_c$ the quotient map. The space $\mathcal{C}(E^u)$ is metrizable, and hence, is a Hausdorff topological space. For every $a \in H_1(\mathbf{T}^2, \mathbf{Z})$, we let $\mathcal{C}_a(E^u) = \pi_c(\widetilde{\mathcal{C}}(E^u) \cap a)$.

**Theorem A** (The invariance of cohomology)**.** *Let $f_0$ and $f_1$ be $C^2$ non-degenerate orientable* $\mathbf{P}$A *diffeomorphisms on $\mathbf{T}^2$, $E^u_i \oplus E^s_i$ the* $\mathbf{P}$A *splitting associated to $f_i$ for each $i = 0, 1$, and $a \in H_1(\mathbf{T}^2, \mathbf{Z})$ a prime homology class. Suppose that $f_0$ and $f_1$ are* $\mathbf{P}$A *homotopic. Then, the pairs $(\mathcal{C}_a(E^u_0) \cup \{\infty\}, \infty)$ and $(\mathcal{C}_a(E^u_1) \cup \{\infty\}, \infty)$ are homotopy equivalent. In particular, the compactly supported cohomology groups $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u_0))$ and $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u_1))$ are isomorphic.*

A $\mathbf{P}$A-diffeomorphism $f$ induces a natural action $\mathcal{C}(f)$ on the space $\mathcal{C}(E^u)$ by $\mathcal{C}(f)(\pi_c(\gamma)) = \pi_c(f \circ \gamma)$. We define the index $\mathrm{ind}\, c$ of a periodic point $c = \pi_c(\gamma)$ of $\mathcal{C}(f)$ by the number of $t \in S^1$ such that $\gamma(t)$ is a repelling periodic point. It is easy to see that the index of $c$ does not depend on the choice of $\gamma \in \pi_c^{-1}(c)$.

**Theorem B** (The Morse decomposition)**.** *Let $f$ be a $C^2$ non-degenerate orientable* $\mathbf{P}$A *diffeomorphism on $\mathbf{T}^2$, $TM = E^u \oplus E^s$ the* $\mathbf{P}$A *splitting associated to $f$, and $a \in H_1(\mathbf{T}^2, \mathbf{Z})$ a prime homology class. Then, $W^u(c; \mathcal{C}(f))$ is homeomorphic to the open disk of dimension* $\mathrm{ind}\, c$ *for all $c \in \mathrm{Per}(\mathcal{C}(f))$ and the decomposition*

$$\mathcal{C}_a(E^u) = \bigcup W^u(c; \mathcal{C}(f))$$

*gives a CW complex structure on $\mathcal{C}_a(E^u)$, where* $\mathrm{Per}(\mathcal{C}(f))$ *denotes the set of periodic points of $\mathcal{C}(f)$, $W^u(c; \mathcal{C}(f))$ denotes the unstable set of $c$ for $\mathcal{C}(f)$, and the union runs over $\mathcal{C} \in \mathrm{Per}(\mathcal{C}(f)) \cap \mathcal{C}_a(E^u)$.*

In Section 2, we give some examples of $\mathbf{P}$A diffeomorphisms on $\mathbf{T}^2$ and calculate the invariants. Section 3 is devoted to the outline of proofs of Theorem A and Theorem B. The details of the proofs and some applications of the invariants will be given in [1].

To end this section, we pose a question. As we see in Section 2, if $f$ is an Anosov diffeomorphism, then $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u_f))$ vanishes for all $a \in H_1(\mathbf{T}^2, \mathbf{Z})$. By Theorem A, $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u_g))$ vanishes for any $C^2$ non-degenerate $\mathbf{P}$A diffeomorphism $g$ which is $\mathbf{P}$A homotopic to $f$. Our question is the converse of this observation.

**Question.** *If $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u_f))$ vanishes for all $a \in H_1(\mathbf{T}^2, \mathbf{Z})$, then is the map $f$* $\mathbf{P}$A *homotopic to an Anosov diffeomorphism?*

## 2. Examples.

**2.1. Anosov diffeomorphisms.** Let $f$ be an Anosov diffeomorphism on $\mathbf{T}^2$ and $TM = E^u \oplus E^s$ the Anosov splitting. It is easy to see that $E^u \oplus E^s$ is the $\mathbf{P}$A splitting, and hence, $f$ is a $\mathbf{P}$A diffeomorphism. Since no circle tangent to $E^u$ exists, we obtain that $\mathcal{C}_a(E^u) = \emptyset$ for all $a \in H_1(\mathbf{T}^2; \mathbf{Z})$.

**2.2. Eliashberg-Thurston's example.** The following example is given by Eliashberg and Thurston [2].

We identify $\mathbf{T}^2$ with $(\mathbf{R}^2 \setminus \{0\})/((x, y) \sim (2x, 2y))$. Fix $\alpha > 1$. We define a diffeomorphism $f_\alpha$ on $\mathbf{T}^2$ by $f_\alpha(x, y) = (\alpha x, \alpha^{-1} y)$ for all $(x, y) \in \mathbf{T}^2$. Let $E^u$ and $E^s$ be the subbundles of $\mathbf{T}^2$ parallel to the $x$-axis and the $y$-axis respectively. It is easy to see that $E^u \oplus E^s$ is the $\mathbf{P}$A splitting associated to $f_\alpha$.

Note that $\mathrm{Per}(f_2)$ is the union of four circles given by the $x$-axis and the $y$-axis and $\mathrm{Per}(f_\alpha)$ is empty if $\alpha^n \notin \mathbf{Z}$ for all $n$. It implies that $f_\alpha$ changes its dynamical properties at $\alpha = 2$. In particular, the dynamics is not invariant under $\mathbf{P}$A homotopy.

Fix $\alpha > 1$ so that $\alpha^n \notin \mathbf{Z}$ for all $n$. Since $\mathrm{Per}(f_\alpha)$ is empty, the map $f_\alpha$ is non-degenerate. We define two $C^1$ maps $\gamma$ and $\gamma'$ from $S^1 = \mathbf{R}/\mathbf{Z}$ to $\mathbf{T}^2$ by $\gamma(t) = (2^t, 0)$ and $\gamma'(t) = (-2^t, 0)$. Let $a \in H_1(\mathbf{T}^2, \mathbf{Z})$ be the homology class represented by $\gamma$. It is easy to see that $\mathcal{C}_a(E^u)$ consists of two index zero fixed points $\pi_c(\gamma)$ and $\pi_c(\gamma')$ of $\mathcal{C}(f_\alpha)$. Hence, we obtain that $H^0_{\mathrm{C}}(\mathcal{C}_a(E^u)) \cong \mathbf{Z}^2$ and $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u)) \cong \{0\}$ for $* \neq 0$. We could also see that $\mathcal{C}_{a'}(E^u) = \emptyset$, and hence, $H^*_{\mathrm{C}}(\mathcal{C}_{a'}(E^u)) \cong \{0\}$ for all $a' \neq \pm a$.

**2.3. Noda's example.** The following is a small modification of the example given by Noda [5].

Define two contact forms $\alpha$ and $\beta$ on the three dimensional torus $\mathbf{R}^3/(2\pi\mathbf{Z})^3$ by $\alpha(x, y, z) = dx + \epsilon \sin x \cdot dy - \cos x \cdot dz$ and $\beta(x, y, z) = dy + \epsilon \sin y \cdot dx - \cos y \cdot dz$. The vector field $v$ given by $v_\epsilon(x, y, z) = (\sin x - \epsilon \cos x \sin y, \sin y - \epsilon \cos y \sin x, 1 - \epsilon^2 \cos x \cos y)$ is contained in $\mathrm{Ker}\, \alpha \cap \mathrm{Ker}\, \beta$. Note that the flow generated by $v_\epsilon$ has a cross section $\mathbf{T}^2 \times \{0\}$ for any $\epsilon \in (-1, 1)$. Since $(\mathrm{Ker}\, \alpha, \mathrm{Ker}\, \beta)$ is a bi-contact structure for any $\epsilon \in (0, 1)$, the result of Mitsumatsu [4] and Eliashberg and Thurston [2] implies that the flow is projectively

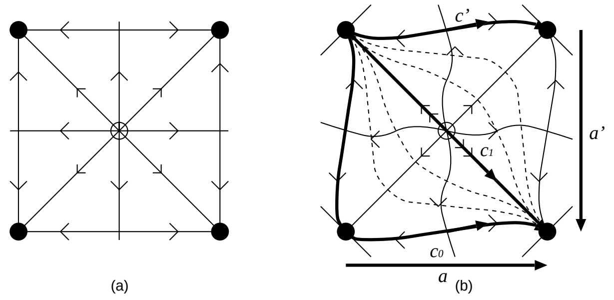


Fig. 1. Phase portrait of $g_0$ and $g_\epsilon$.

Anosov, and hence, the return map $g_\epsilon$ on $\mathbf{T}^2 \times \{0\}$ is a $\mathbf{P}$A diffeomorphism.

Since $v_0(x, y, z) = (-\sin x, -\sin y, 1)$, the return map $g_0$ has the phase portrait as in Fig. 1 (a). Hence, we obtain the phase portrait of $g_\epsilon$, which is a perturbation of $g_0$, as in Fig. 1 (b).

In the Fig. 1, black balls and a white ball mean an attracting fixed point and a repelling fixed point respectively. Let $c_0$, $c_1$ and $c'$ are the elements of $\mathcal{C}(E^u)$ and $a$ and $a'$ are homology class given in Fig. 1. It is easy to see that $\mathrm{Per}(\mathcal{C}_a(E^u))$ consists of the index zero fixed point $c'$. By Theorem B, we obtain that $\mathcal{C}_a(E^u) = \{c'\}$. In particular, $H^0_{\mathrm{C}}(\mathcal{C}_a(E^u)) \cong \mathbf{Z}$ and $H^*_{\mathrm{C}}(\mathcal{C}_a(E^u)) \cong \{0\}$ for any $* \neq 0$. It is also easy to see that $\mathrm{Per}(\mathcal{C}_{a+a'}(g_\epsilon))$ consists of $c_0$ and $c_1$ and $\mathrm{ind}\, c_i = i$ for each $i = 0, 1$. By Theorem B, the space $\mathcal{C}'_{a+a}(E^u)$ is a CW complex with a one dimensional cell and a zero dimensional cell. Hence, $\mathcal{C}_{a+a'}(E^u)$ is homeomorphic to $S^1$. In particular, $H^*_{\mathrm{C}}(\mathcal{C}_{a+a'}(E^u)) \cong \mathbf{Z}$ for $* = 0, 1$ and $H^*_{\mathrm{C}}(\mathcal{C}_{a+a'}(E^u)) \cong \{0\}$ otherwise.

Define a covering map $\pi_{m,n}$ on $\mathbf{T}^2$ by $\pi_{m,n}(x, y) = (mx, ny)$ for every $m, n \geq 1$. Let $g_{m,n}$ be a lift of $g_\epsilon$ by $\pi_{m,n}$ and $E^u_{m,n} \oplus E^s_{m,n}$ the $\mathbf{P}$A splitting associated to $g_{m,n}$. We can see that $\mathcal{C}_a(E^u_{m,n})$ and $\mathcal{C}_{a'}(E^u_{m,n})$ consist of $n$ and $m$ elements of index 0 respectively. Hence, we obtain that $H^0_{\mathrm{C}}(\mathcal{C}_a(E^u_{m,n})) \cong \mathbf{Z}^n$ and $H^0_{\mathrm{C}}(\mathcal{C}_{a'}(E^u_{m,n})) \cong \mathbf{Z}^m$. By Theorem A, the maps $g_{m,n}$ and $g_{m',n'}$ are $\mathbf{P}$A-homotopic if and only if $(m, n) = (m', n')$. However, all $E^u_{m,n}$ are homotopic as plane fields.

## 3. Outline of proof.

**3.1. Theorem B.** First, we give the outline of the proof of Theorem B. The following is a deep result of Pujals and Sambarino on surface dynamics.

**Proposition 3.1** (Pujals-Sambarino, [6]). *Let $f$ be a $C^2$ non-degenerate diffeomorphism on a compact surface $M$. Suppose that there exists a dominated splitting associated to $f$ on the non-wandering set $\Omega(f)$. Then, there exists a disjoint decomposition $\Omega(f) = \Omega_0(f) \sqcup \Omega_1(f) \sqcup \Omega_2(f)$ such that*

1. *$\Omega_1(f)$ is a hyperbolic set of saddle-type,*
2. *$\Omega_0(f)$ is the union of finite number of attracting periodic orbits and normally attracting embedded circles with no periodic points, and*
3. *$\Omega_2(f)$ is the union of finite number of repelling periodic orbits and normally repelling embedded circles with no periodic points.*

Let $\mathrm{Per}^k_h(f)$ be the set of all hyperbolic periodic points with the $k$ dimensional unstable manifold. The following is a consequence of Proposition 3.1 and a variation of the $C^1$ regularity theorem for codimension one hyperbolic splittings.

**Proposition 3.2.** *The subbundle $E^u$ defines $C^1$ foliation $\mathcal{F}^u$ on $\mathbf{T}^2 \setminus \Omega_0(f)$. Moreover, $\mathcal{F}^u$ contains no closed leaves and if a leaf $\mathcal{F}^u(z)$ has finite length then the boundary $\partial\mathcal{F}^u(z)$ is contained in $\mathrm{Per}^0_h(f)$.*

For every $f \in \mathbf{P}\mathrm{A}^1(\mathbf{T}^2)$, we say a pair $(e^u, e^s)$ of non-singular continuous vector fields is an *orientation* of $f$ when $\mathbf{R}e^u \oplus \mathbf{R}e^s$ is the $\mathbf{P}$A splitting associated to $f$. We call a triple $(f, e^u, e^s)$ an *oriented* $\mathbf{P}$A diffeomorphism on $\mathbf{T}^2$ if $(e^u, e^s)$ is an orientation of $f$.

Fix $(f, e^u, e^s) \in \mathbf{P}\mathrm{A}^2(\mathbf{T}^2)$ such that $f$ is non-degenerate. We define the space $\mathcal{S}(e^u)$ by

$$\mathcal{S}(e^u) = \big\{\xi \in C^1([0,1], \mathbf{T}^2) \;\big|\; \dot{\xi}(t) = a \cdot e^u(\xi(t)) \text{ for some } a > 0\big\}$$

with $C^1$-topology. Remark that the space $\mathcal{S}(e^u)$ is locally equi-continuous. Hence, it is locally compact by the Ascoli-Arzelá theorem. The space $\mathcal{S}(e^u)$ has a natural composition $*$ of two paths and a natural action $\mathcal{S}(f)$ of $f$.

For every compact subset $\Lambda$ of $\mathbf{T}^2$, let $\mathcal{S}(e^u, \Lambda)$ be the subset of $\mathcal{S}(e^u)$ consisting of the elements $\xi$ such that $\xi(0), \xi(1) \in \Lambda$. By Proposition 3.2, if $\mathcal{F}^u(z)$ has finite length for a point $z \in \mathbf{T}^2 \setminus \Omega_0(f)$ then there exists the unique element $(z)_u$ of $\mathcal{S}(e^u, \mathrm{Per}^0_h(f))$ such that $(z)_u((0,1)) = \mathcal{F}^u(z)$. The followings are the key lemmas to show Theorem B.

**Lemma 3.3.** *For every $\xi \in \mathcal{S}(e^u, \Omega_0(f))$ there exist a sequence $\{z_i\}$ in $\mathbf{T}^2 \setminus \Omega_0(f)$ and a sequence $\{p_i\}$ in $\mathrm{Per}^1_h(f) \cup \mathrm{Per}^2_h(f)$ such that $\xi = (z_1)_u * \cdots * (z_n)_u$ and $(z_i)_u \in W^u(p_i; \mathcal{S}(f))$. In particular, $\xi \in W^u((p_1)_u * \cdots * (p_n)_u; \mathcal{S}(f))$.*

**Lemma 3.4.** *Let $k = 1, 2$ and $p \in \mathrm{Per}_h^k(f)$. Suppose that $W^u(p; f)$ has finite length. Then, $(z)_u \in W^u((p)_u; \mathcal{S}(f))$ if and only if $z \in W^u(p; f)$. Moreover, in the case of $k = 2$, there exists a continuous map $\mu_p$ from $[-1, 1]$ to $\mathcal{S}(e^u, \Omega_0(f))$ such that $\mu_p(0) = (p)_u$, $\mu_p|_{(-1,1)}$ is a homeomorphism between $(-1, 1)$ and $W^u((p)_u; \mathcal{S}(f))$.*

The above lemmas imply the existence of a CW complex structure on $\mathcal{S}(e^u, \mathrm{Per}_h^0(f))$.

**Proposition 3.5.** *The space $\mathcal{S}(e^u, \mathrm{Per}_h^0(f))$ has a CW complex structure*

$$\mathcal{S}(e^u, \mathrm{Per}_h^0(f)) = \bigcup W^u(\xi_0; \mathcal{S}(f)),$$

*where the union runs over*

$$\xi_0 \in \mathrm{Per}(\mathcal{S}(f)) \cap \mathcal{S}(e^u, \mathrm{Per}_h^0(f)).$$

*Moreover, the dimension of $W^u(\xi_0; \mathcal{S}(f))$ is equal to the number of $t \in S^1$ such that $\xi_0(t) \in \mathrm{Per}_h^2(f)$.*

Let $\mathcal{S}_c(e^u, \Lambda)$ be the subset of $\mathcal{S}(e^u, \Lambda)$ consisting of the elements such that $\xi(0) = \xi(1)$. By Propositions 3.1 and 3.2, every $c = \pi_c(\gamma) \in \mathcal{C}(E^u)$ satisfies that either $\mathrm{Im}\,\gamma$ is an attracting embedded circle with an irrational rotation or $\mathrm{Im}\,\gamma$ contains attracting periodic points. In the former case, we can show that $c$ is periodic, has the index zero and is isolated in $\mathcal{C}(E^u)$. In the latter case, $c$ is contained in $\pi_c(\mathcal{S}_c(\pm e^u, \mathrm{Per}_h^0(f)))$. When $a \in H_1(\mathbf{T}^2; \mathbf{Z})$ is prime, $\pi_c$ is injective on $W^u(\xi_0; \mathcal{S}(f))$ for each $\xi_0 \in \mathrm{Per}(\mathcal{S}(f)) \cap \mathcal{S}_c(\pm e^u, \mathrm{Per}_h^0(f)) \cap a$. Hence, the CW complex structure of $\mathcal{S}_c(\pm e^u, \mathrm{Per}_h^0(f)) \cap a$ induce that of $\mathcal{C}_a(E^u)$. It completes the proof of Theorem B.

**3.2. Theorem A.** Let $\{f_t\}_{t\in[0,1]}$ be a $\mathbf{P}$A homotopy. For $t_* \in (0, 1)$ and $N \geq 1$, we say a fixed point $p_*$ of $f_{t_*}^N$ *exhibits a positive saddle-node bifurcation* when there exist a neighborhood $U$ of $p_*$, a positive number $\delta$, and two continuous functions $p_s$, $p_u$ from $[t_*, t_* + \delta]$ to $U$ such that $p_s(t_*) = p_n(t_*) = p_*$ is the unique fixed point of $f_t^N$ in $U$, there exist no fixed points of $f_t^N$ in $U$ for all $t \in (t_* - \delta, t_*)$, and $p_s(t) \in \mathrm{Per}_h^1(f_t)$ and $p_n(t) \in \mathrm{Per}_h^0(f_t) \cup \mathrm{Per}_h^2(f_t)$ are the only fixed points in $U$ for all $t \in (t_*, t_* + \delta)$.

A $\mathbf{P}$A homotopy $\{f_t\}_{t\in[0,1]}$ is called *regular* if $f_0$, $f_1$ are non-degenerate, $f_t(z)$ is $C^2$ with respect to $(z, t)$, $\mathrm{Per}(f_t)$ contains at most one non-hyperbolic periodic orbit, and it exhibits a saddle-node bifurcation. Since $\mathbf{P}\mathrm{A}^r(\mathbf{T}^2)$ is open in $\mathrm{Diff}^r(\mathbf{T}^2)$ for all $r$ and a saddle-node bifurcation is the only generic bifurcation that an orientable $\mathbf{P}$A diffeomorphism exhibits, every $\mathbf{P}$A homotopy connecting two $C^2$ non-degenerate orientable $\mathbf{P}$A diffeomorphisms can be approximated by a regular $\mathbf{P}$A homotopy. In particular, in order to show Theorem A, we have only to consider regular $\mathbf{P}$A homotopies.

For every $\Lambda \subset \mathbf{T}^2$, let $\mathcal{C}(E^u, \Lambda)$ be the set of all $\pi_c(\gamma) \in \mathcal{C}(E^u)$ such that $\mathrm{Im}\,\gamma \cap \Lambda \neq \emptyset$. Let $\Lambda_{\mathrm{sing}}^u(f)$ denote the union of all $f$-periodic immersed circle tangent to $E_f^u$ which contains no hyperbolic periodic points. We say a $\mathbf{P}$A homotopy $\{f_t\}_{t\in[0,1]}$ is *a simple* $\mathbf{P}$A *homotopy* when

1. there exist a regular $\mathbf{P}$A homotopy $\{g_t\}$ and an integer $N \geq 1$ such that $f_t = g_t^N$ for all $t \in [0, 1]$,
2. there exists $t_* \in (0, 1)$ such that all fixed points of $f_t$ are hyperbolic for any $t \neq t_*$,
3. any non-hyperbolic fixed point of $f_{t_*}$ exhibit a positive saddle-node bifurcation,
4. $\mathcal{C}(E_{t_*}^u, \mathrm{Per}(f_{t_*})) \cap \mathrm{Per}(\mathcal{C}(f)) = \mathcal{C}(E_{t_*}^u, \mathrm{fix}(f_{t_*})) \cap \mathrm{fix}(\mathcal{C}(f))$, and
5. there exists the unique continuation $\Lambda(t)$ of $\Lambda_{\mathrm{sing}}^u(f_{t_*})$ on $[0, 1]$.

**Proposition 3.6.** *Let $\{f_t\}_{t\in[0,1]}$ be a regular* $\mathbf{P}$A *homotopy. For every $t_0 \in [0, 1]$, there exist a neighborhood $I$ of $t_0$, a homeomorphism $h$ from $[0, 1]$ to $I$, and an integer $N \geq 1$ such that $\{f_{h(t)}^N\}_{t\in[0,1]}$ is a simple* $\mathbf{P}$A *homotopy.*

The proof is given by showing the boundedness of the period of periodic embedded circles. By the above proposition, we have only to consider simple $\mathbf{P}$A homotopies.

**Proposition 3.7.** *Let $\{f_t\}_{t\in[0,1]}$ be a simple* $\mathbf{P}$A *homotopy and $E_t^u \oplus E_t^s$ the* $\mathbf{P}$A *splitting associated to $f_t$. Then, $(\mathcal{C}_a(E_0^u) \cup \{\infty\}, \infty)$ and $(\mathcal{C}_a(E_1^u) \cup \{\infty\}, \infty)$ are homotopic for every prime homology class $a \in H_1(\mathbf{T}^2; \mathbf{Z})$.*

With a careful investigation of the continuation of fixed points of $\mathcal{C}(f_{t_*})$, we obtain the information on changing of the CW complex structures in Theorem B under a simple homotopy. It makes us enable to check the homotopy equivalence.

Theorem A is an immediate consequence of Propositions 3.6 and 3.7.

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## References

[ 1 ] Asaoka, M.: Invariants of two dimensional projectively Anosov diffeomorphisms and their applications. (Preprint).

[ 2 ] Eliashberg, Y., and Thurston, W.: Confoliations. Amer. Math. Soc., Providence, RI (1998).

[ 3 ] Katok, A., and Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems. Encyclopedia of Math. and its Appl., vol. 54, Cambridge Univ. Press, Cambridge (1995).

[ 4 ] Mitsumatsu, Y.: Anosov flows and non-Stein symplectic manifolds. Ann. Inst. Fourier, **45**(5), 1407–1421 (1995).

[ 5 ] Noda, T.: Personal communication.

[ 6 ] Pujals, E., and Sambarino, M.: Homoclinic tangencies and hyperbolicity for surfaces diffeomorphisms. Ann. of Math., **151**, 961–1023 (2000).