A modular group action on cubic surfaces and the monodromy of the Painlevé VI equation

By Katsunori Iwasaki

Faculty of Mathematics, Kyushu University, 6-10-1, Hakozaki, Higashi-ku, Fukuoka 812-8581
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Abstract: We construct an action of the modular group $\Gamma(2)$ on a general 4-parameter family of complex cubic surfaces and describe the nonlinear monodromy of the Painlevé VI equation in terms of this action.

Key words: Modular group; complex cubic surface; nonlinear monodromy; the Painlevé VI equation.

1. Introduction. In this paper we construct an action of the modular group $\Gamma(2)$ of level 2 on a general 4-parameter family of complex cubic surfaces and describe the nonlinear monodromy of the Painlevé VI equation in terms of this action. Here the Painlevé VI equation $\mathrm{P_{VI}} = \mathrm{P_{VI}}(\kappa) = \mathrm{P_{VI}}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ is a second order nonlinear ordinary differential equation

$$\begin{aligned} y_{xx} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y_x^2 \\ &\quad - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y \\ &\quad + \frac{y(y-1)(y-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{x}{y^2}\right. \\ &\qquad \left. + \kappa_2^2 \frac{x-1}{(y-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(y-x)^2}\right\} \end{aligned}$$

with complex parameters $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbf{C}^4$. See Iwasaki *et al.* [7] for general information about it.

Discussing monodromy or analytic continuations of solutions is a fundamental problem for $\mathrm{P_{VI}}$. From many previous works in this direction we cite a series of papers [4, 8, 9] by Dubrovin and Mazzocco. The first paper [4] deals with a special 1-parameter family $\mathrm{P_{VI}}(\mu)$ with $\kappa_1 = \kappa_2 = \kappa_3 = 0$ and $\kappa_4 = 2\mu - 1 \notin \mathbf{Z}$ (the non-resonant case). It gives a description of monodromy and the classification of algebraic solutions in the family (5 solutions). The second paper [8] treats the same equation in the resonant case $2\mu - 1 \in \mathbf{Z}$, which involves the classical Picard and Chazy solutions [14, 3]. Evidently their work should be generalized to the full 4-parameter family. In this direction, Mazzocco [9] classifies the rational solutions for the full 4-parameter family. The paper [6] by Guzzetti, who settles the connection problem for the general $\mathrm{P_{VI}}$, should also be cited here.

In this paper we take up the problem of describing the monodromy explicitly in terms of certain natural coordinates, since such a description seems to have many applications. After constructing the monodromy as a modular group action on cubic surfaces, we will discuss some applications, pose some questions and problems and indicate future directions that should follow our construction. We insist on the viewpoint of studying the monodromy as a complex dynamical system on cubic surfaces.

This paper is just an announcement and a full account of the results together with some further materials will be presented separately.

2. A Modular group action. Let $\mathbf{C}^7 = \mathbf{C}^3 \times \mathbf{C}^4$ be the complex 7-space with coordinates (x, a), where $x = (x_1, x_2, x_3) \in \mathbf{C}^3$ is space variables and $a = (a_1, a_2, a_3, a_4) \in \mathbf{C}^4$ is parameters. Throughout the paper we denote by (i, j, k) any cyclic permutation of $(1, 2, 3)$ and put

$$\theta_i(a) = a_i a_4 + a_j a_k.$$

Let $g_i : (x, a) \mapsto (x', a')$ be a polynomial automorphism of $\mathbf{C}^7$ defined by

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$$(1)\qquad g_i : \begin{cases} x_i' = \theta_j(a) - x_j - x_k x_i, \\ x_j' = x_i, \\ x_k' = x_k, \\ a_i' = a_j, \\ a_j' = a_i, \\ a_k' = a_k, \\ a_4' = a_4. \end{cases}$$

Let $G = \langle g_1, g_2, g_3 \rangle$ be the group generated by the transformations g_1, g_2, g_3. Then a direct check shows that the generators satisfy three relations

$$g_i g_j g_i = g_j g_i g_j, \quad (g_i g_j)^3 = 1, \quad g_k = g_i g_j g_i^{-1}.$$

The last relation implies that G is generated by two elements g_i and g_j, while the first two tell us that

$$s = g_i g_j g_i, \qquad t = g_i,$$

satisfy the well-known defining relations of the full modular group $\Gamma = PSL(2, \mathbf{Z})$, namely,

$$s^2 = (ts)^3 = 1.$$

Conversely, under these relations, one has

$$g_i = t, \qquad g_j = sts, \qquad g_k = st^{-2}.$$

Hence G is generated by the elements s, t and there exists a surjective group homomorphism $\Gamma \to G$ such that $S \mapsto s$ and $T \mapsto t$, where

$$S(z) = -1/z, \qquad T(z) = z + 1$$

are the standard generators of Γ. Through this homomorphism, the modular group Γ acts on $\mathbf{C}^7$ as a polynomial automorphism group. Let $\Gamma(2)$ be the principal congruence subgroup of Γ of level 2. It is easy to see that the image of $\Gamma(2)$ under the homomorphism $\Gamma \to G$ coincides with the subgroup $G(2) = \langle g_1^2, g_2^2, g_3^2 \rangle$ of G generated by g_1^2, g_2^2, g_3^2.

There exists a natural surjective homomorphism $G \to S_3$ sending g_i to the substitution $\sigma_i = (i, j)$, where the symmetric group S_3 acts on the parameters $a \in \mathbf{C}^4$ by permuting the first three coordinates (a_1, a_2, a_3). The kernel of the homomorphism $G \to S_3$ is exactly the subgroup $G(2)$. For each element $g \in G$, there exists a commutative diagram

$$(2)\qquad \begin{array}{ccc} \mathbf{C}^7 & \xrightarrow{\;g\;} & \mathbf{C}^7 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbf{C}^4 & \xrightarrow[\sigma]{} & \mathbf{C}^4. \end{array}$$

where $\pi : (x, a) \mapsto a$ is the projection down to parameters and $\sigma \in S_3$ is the permutation corresponding to the element $g \in G$. Since $\sigma = 1$ for $g \in G(2)$, the subgroup $\Gamma(2)$ acts on $\mathbf{C}^7$ keeping the parameters a invariant. We thus have a 4-parameter family of $\Gamma(2)$-actions on $\mathbf{C}^3$ parametrized by $a \in \mathbf{C}^4$.

3. A family of cubic surfaces. It is an important observation that the polynomial

$$\begin{aligned} f(x, a) = {} & x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 \\ & - \theta_1(a)\, x_1 - \theta_2(a)\, x_2 - \theta_3(a)\, x_3 \\ & + a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 \end{aligned}$$

is G-invariant. Hence each level set of the function f is kept invariant under the action of G. In this paper we are particularly interested in the zero level set

$$\mathcal{S} = \{\, (x, a) \in \mathbf{C}^7 \;:\; f(x, a) = 0 \,\},$$

since it is closely related to the solution space of the Painlevé VI equation. Restricting the projection $\pi : \mathbf{C}^7 \to \mathbf{C}^4$ to $\mathcal{S}$ yields a fibration $\pi : \mathcal{S} \to \mathbf{C}^4$ and then (2) induces a commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\;g\;} & \mathcal{S} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbf{C}^4 & \xrightarrow[\sigma]{} & \mathbf{C}^4. \end{array}$$

The fiber of $\pi : \mathcal{S} \to \mathbf{C}^4$ over each $a \in \mathbf{C}^4$ is an affine complex cubic surface

$$\mathcal{S}(a) = \{\, x \in \mathbf{C}^3 \;:\; f_a(x) = 0 \,\},$$

where we write $f_a(x) = f(x, a)$ regarding it as a polynomial of x depending on parameters a. Thus $\mathcal{S}$ defines a 4-parameter family of affine cubic surfaces. From the construction above, we have an action of the modular group Γ on the family $\mathcal{S}$, which restricts to an action of $\Gamma(2)$ on each cubic surface $\mathcal{S}(a)$.

It is well known in classical algebraic geometry that the isomorphism classes of cubic surfaces in the complex projective 3-space have a 4-dimensional moduli space and that Cayley [2] constructed a 4-parameter family of general cubic surfaces, known nowadays as Cayley's normal form. Recently (as compared with Cayley's era), Naruki and Sekiguchi [11, 12] modified Cayley's normal form in a convenient way. A direct calculation shows that our family $\mathcal{S}$ and Cayley's normal form (modified by Naruki and Sekiguchi) have a common algebraic cover and hence our family captures general moduli. Thus we can propose $\mathcal{S}$ as another normal form than Cayley's.

It is of interest to find out a necessary and sufficient condition on $a \in \mathbf{C}^4$ under which the cubic surface $\mathcal{S}(a)$ becomes (non)singular.

Theorem 1. *Let $w(a)$ be a polynomial of $a = (a_1, a_2, a_3, a_4)$ defined by*

$$w(a) = \prod_{\epsilon_1\epsilon_2\epsilon_3=1} (\epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3 + a_4) - \prod_{i=1}^{3} (a_i a_4 - a_j a_k), \tag{3}$$

where the first product on the right-hand side is taken over all triples $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3$ satisfying $\epsilon_1\epsilon_2\epsilon_3 = 1$. Then the affine cubic surface $\mathcal{S}(a)$ has singular points if and only if

$$w(a) \prod_{i=1}^{4} (a_i^2 - 4) = 0. \tag{4}$$

Proof. The gradient vector field $\operatorname{grad} f_a$ of the function f_a is expressed as $y = (y_1, y_2, y_3)$, where $y_i = y_i(x, a)$ is given by

$$y_i(x, a) = 2x_i + x_j x_k - \theta_i(a). \tag{5}$$

Hence a point $x \in \mathcal{S}(a)$ is a singular point if and only if $y = 0$. For $m = 1, 2, 3$, let E_m be the m-th elementary symmetric polynomial of $x = (x_1, x_2, x_3)$ and similarly let e_m be the m-th elementary symmetric polynomial of $\theta = (\theta_1, \theta_2, \theta_3)$. Under the condition $y = 0$, we have

$$e_1 = E_2 + 2E_1, \tag{6}$$

$$e_2 = E_1 E_3 - 6E_3 + 2E_1 E_2 + 4E_2, \tag{7}$$

$$e_3 = E_3^2 - 4E_2 E_3 + 2E_1^2 E_3 - 8E_1 E_3 + 8E_3 + 4E_2^2. \tag{8}$$

On the other hand, equation $f_a = 0$ is expressed as

$$e_4 = 2E_3 - 2E_2 + E_1^2, \tag{9}$$

where we set $e_4 = a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4$. From (6) and (9) we have

$$E_2 = e_1 - 2E_1,$$
$$2E_3 = 2e_1 + e_4 - 4E_1 - E_1^2.$$

Substituting these into (7) and (8), we obtain

$$\begin{cases} E_1^3 + 6E_1^2 - (6e_1 + e_4 + 8)E_1 \\ \qquad + 4e_1 + 2e_2 + 6e_4 = 0, \\ 3E_1^4 + 8E_1^3 - (12e_1 + 2e_4 + 64)E_1^2 \\ \quad + (48e_1 + 8e_4 + 64)E_1 - 32e_1 - 4e_1^2 \\ \quad + 4e_3 - 16e_4 + 4e_1 e_4 - e_4^2 = 0. \end{cases} \tag{10}$$

The resultant of these two algebraic equations is

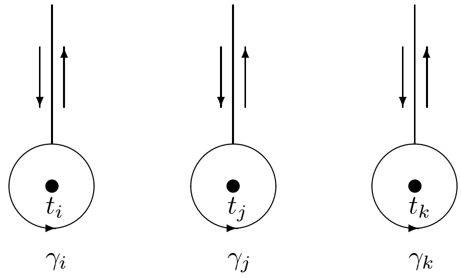

Fig. 1. The loops γ_i, γ_j, γ_k.

$$-16w^2(a) \prod_{i=1}^{4} (a_i^2 - 4),$$

where $w(a)$ is defined by (3). The cubic surface $\mathcal{S}(a)$ is singular if and only if equations (10) have common roots; this is the case if and only if the resultant vanishes. Therefore the theorem is established. □

Problem 2. What is the polynomial $w(a)$? Does it have anything to do with modular functions?

4. Monodromy of the Painlevé VI equation. The Painlevé VI equation is characterized as the isomonodromic deformation equation of second-order Fuchsian differential equations with four regular singular points and an apparent singular point on the Riemann sphere $\mathbf{P}^1$ [5]. It is also obtained as a symmetry reduction of the rank two Schlesinger system [16], which describes the isomonodromic deformation of rank two Fuchsian systems with four regular singular points on $\mathbf{P}^1$. Here one singular point is fixed at infinity and the remaining three singular points, say, t_1, t_2, t_3, play the role of independent variables of the Schlesinger system.

The isomonodromic nature implies that a solution germ (at a base point) of the Schlesinger system can be identified with a monodromy data, that is, a triple $M = (M_1, M_2, M_3) \in SL(2, \mathbf{C})^3$ of monodromy matrices of the Fuchsian system, where M_i denotes the monodromy matrix along the loop γ_i indicated in Fig. 1. Through the reduction relative to the adjoint action, a solution germ (at a base point) of the Painlevé VI equation can be identified with a conjugacy class of monodromy data. Here two monodromy data $M = (M_1, M_2, M_3)$ and $M' = (M_1', M_2', M_3')$ are said to be conjugate if there exists a matrix $P \in SL(2, \mathbf{C})$ such that $M_i' = P M_i P^{-1}$ for $i = 1, 2, 3$. Hereafter a conjugacy class of monodromy data is simply called a monodromy data and the conjugacy class containing a data $M =$

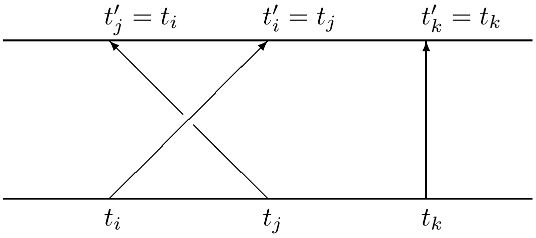

Fig. 2. The braid β_i.

(M_1, M_2, M_3) is denoted by the same symbol M.

As is mentioned in [4], the nonlinear monodromy of the Painlevé VI equation is described in terms of an action of braids on monodromy data. Fix three distinct points $t_1, t_2, t_3 \in \mathbf{C}$ and let B_3 be the braid group on three strings with base points t_1, t_2, t_3. The group B_3 is generated by three braids β_1, β_2, β_3, where β_i is indicated in Fig. 2. Note that the generators satisfy relations

$$\beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j, \qquad \beta_k = \beta_i^{-1}\beta_j\beta_i.$$

Each braid $\beta \in B_3$ acts on the loops γ_1, γ_2, γ_3 in a natural manner. The action of β_i is given by

$$\beta_i : \quad \begin{cases} \gamma_i' = \gamma_i\gamma_j\gamma_i^{-1}, \\ \gamma_j' = \gamma_i, \\ \gamma_k' = \gamma_k. \end{cases}$$

Then the isomonodromic deformation induces an action $\beta_i : M \mapsto M'$ on monodromy data in such a manner that the monodromy data M with respect to the loops γ_1, γ_2, γ_3 is identical with the monodromy data M' with respect to the loops γ_1', γ_2', γ_3'. Explicitly, M' is represented as

$$\text{(11)} \qquad \beta_i : \quad \begin{cases} M_i' = M_j, \\ M_j' = M_jM_iM_j^{-1}, \\ M_k' = M_k. \end{cases}$$

Recall that the center $Z(B_3)$ of the braid group B_3 is the cyclic group generated by $(\beta_j\beta_i)^3$ and the quotient group $B_3/Z(B_3)$ is isomorphic to the full modular group Γ (see e.g. Birman [1]). It is easily seen that the B_3-action defined by (11) is trivial when restricted to the center $Z(B_3)$. Hence we have an action of Γ on monodromy data. Restriction of the B_3-action to the pure braid group $P_3 = \langle \beta_1^2, \beta_2^2, \beta_3^2 \rangle$ induces the restriction of the Γ-action above to a $\Gamma(2)$-action, which is none other than the nonlinear monodromy of the Painlevé VI equation.

Let $\mathcal{M}$ be the space of monodromy data and consider a map $\mathcal{M} \to \mathbf{C}^7$, $M \mapsto (x, a)$ defined by

$$\begin{cases} x_i = \mathrm{Tr}(M_jM_k), \\ a_i = \mathrm{Tr}\, M_i, \qquad\qquad (i = 1, 2, 3), \\ a_4 = \mathrm{Tr}(M_3M_2M_1). \end{cases}$$

We can show that $(x, a) = (x_1, x_2, x_3, a_1, a_2, a_3, a_4)$ admits a relation $f(x, a) = 0$, where $f(x, a)$ is the polynomial introduced in Section 3, and hence we have a map $\phi : \mathcal{M} \to \mathcal{S}$. This map is generically one-to-one, that is, there exists a Zariski open subset $\mathcal{T}$ of $\mathcal{S}$ such that $\phi : \phi^{-1}(\mathcal{T}) \to \mathcal{T}$ is bijective. Thus one may expect that the Γ-action constructed in the last paragraph can be written down explicitly in terms of the coordinates (x, a). This is actually the case and the expression we seek is the formula (1). Here the transformation g_i corresponds to the braid β_i. Now the discussions so far yield the following:

Theorem 3. *The nonlinear monodromy of the Painlevé VI equation* $\mathrm{P_{VI}}(\kappa)$ *is represented by the* $\Gamma(2)$*-action on the cubic surface* $\mathcal{S}(a)$ *constructed in Sections* 2 *and* 3*, where* κ *and* a *are related by*

$$\begin{aligned} a &= (a_1, a_2, a_3, a_4) \\ &= (2\cos\pi\kappa_1, 2\cos\pi\kappa_2, 2\cos\pi\kappa_3, 2\cos\pi\kappa_4). \end{aligned}$$

It is of some interest to notice that the parameters $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of the Painlevé VI equation are closely related or almost identical with the moduli parameters $a = (a_1, a_2, a_3, a_4)$ of cubic surfaces.

5. Monodromy as a complex dynamics. Representation of the monodromy as an action of the modular group on complex cubic surfaces prompts a thorough study of the action as a complex dynamical system. Note that investigating the $\Gamma(2)$-action on $\mathcal{S}(a)$ parametrized by $a \in \mathbf{C}^4$ is essentially the same as investigating the Γ-action on $\mathcal{S}$, since $\Gamma(2)$ is a finite-index subgroup of Γ. We now pose the following:

Problem 4. For the action of Γ on $\mathcal{S}$, the following three issues are of fundamental importance.

1. Classification of the fixed points.
2. Classification of the finite orbits.
3. Classification of the bounded orbits.

The first issue seems to be closely related with the Riccati solutions of the Painlevé VI equation, that is, those solutions which can be expressed in terms of Gauss hypergeometric functions. Clearly, the second issue is connected with the classification of algebraic solutions to $\mathrm{P_{VI}}$. In view of the original motivation that led to the discovery of Painlevé equations — find out new transcendental functions satisfied by some *good* nonlinear differential equation — the third issue is also interesting or even the most important among the three. In fact, a solution to P_{VI} having a *bounded*, but not finite, monodromy could be thought of as a transcendental but *tame* solution. This class of solutions should be handled without too much difficulty, while general transcendental solutions might be too transcendental to allow any moderate treatment.

For the first issue, that is, for the description of the fixed points, we have the following:

Theorem 5. *A point in $\mathcal{S}(a)$ is a fixed point of the $\Gamma(2)$-action if and only if it is a singular point of $\mathcal{S}(a)$. Hence Theorem* 1 *implies that $\mathcal{S}(a)$ contains fixed points if and only if $a \in \mathbf{C}^4$ satisfies* (4)*.*

This theorem readily follows from the observation that a point $x \in \mathcal{S}(a)$ is fixed by the action of $\Gamma(2)$ if and only if $g_1^2(x,a) = g_2^2(x,a) = g_3^2(x,a) = (x,a)$, and this latter condition is equivalent to $y = (y_1, y_2, y_3) = 0$, where y_i is defined by (5). For every parameters $a \in \mathbf{C}^4$ satisfying condition (4), the fixed points can be described explicitly.

The investigation of the second issue is now in progress. As for the third issue, we can introduce the concept of filled Julia set $K(a)$ and Julia set $J(a)$ depending on parameters $a \in \mathbf{C}^4$ (cf. Morosawa *et al.* [10]. The filled Julia set $K(a)$ is the set of points $x \in \mathcal{S}(a)$ such that the orbit through x is bounded; we can show that $K(a)$ is compact. Then the Julia set $J(a)$ is defined to be the boundary of $K(a)$. Now a quite interesting problem is the exploration of the topology of $J(a)$ as well as its dependence upon parameters a. Especially we may ask: Can one introduce a concept of Mandelbrot set on the parameter space? These points should be discussed in forthcoming papers.

Our representation of the nonlinear monodromy for P_{VI} has been obtained as a discrete dynamical system on the space of monodromy data. It is also an interesting problem to describe the monodromy as dynamical systems on the spaces of initial conditions constructed by Okamoto [13], Shioda and Takano [17], Saito, Takebe and Terajima [15] etc. and to discuss the relationship between them and ours.

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