Zeta extensions

By Nobushige Kurokawa*) and Masato Wakayama**)

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Abstract: In the papers [KW2, KW3] we introduced and studied a new type of the Selberg zeta function called by a higher Selberg zeta function. We have established the analytic properties, especially the functional equation of the higher Selberg zeta functions in [KW3]. Motivated by this study of higher Selberg zeta functions we formulate the problem for general zeta functions which have the Euler products and discuss their general features.

Key words: Riemann's zeta function; Selberg's zeta function; Euler product; functional equations.

1. Zeta extensions.

Let

$$\varphi(s) = \prod_{p \in P} H_p(N(p)^{-s})^{-1}$$

be a meromorphic zeta function having an Euler product, where $H_p(u)$ is a power series in u. Typical examples are given by the Riemann zeta function $\zeta(s)$ and the Selberg zeta function $Z_\Gamma(s)$. Actually the Riemann zeta function is obtained by setting $N(p) = p$, that is,

$$\zeta(s) = \prod_{p:\text{primes}} (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1. \tag{1.1}$$

and meromorphic in the whole $\mathbf{C}$.

We next define the Selberg zeta function $Z_\Gamma(s)$ as follows: Let Γ be a discrete, torsion-free subgroup of $SL_2(\mathbf{R})$ with a co-finite volume. Then Γ acts discontinuously on the upper half plane H. Let $\mathrm{Prim}(\Gamma)$ be the set of hyperbolic primitive conjugacy classes of Γ, where a hyperbolic element P (and hence also its conjugacy class) is said to be primitive when P is a generator of an infinite cyclic group $\mathcal{Z}_\Gamma(P)$, the centralizer of P in Γ. Then we define $Z_\Gamma(s)$ by

(1.2)

$$Z_\Gamma(s) = \prod_{n=0}^{\infty} \prod_{P \in \mathrm{Prim}(\Gamma)} (1 - N(P)^{-s-n}), \quad \mathrm{Re}(s) > 1.$$

Here the norm $N(P)\,(>1)$ of $P \in \mathrm{Prim}(\Gamma)$ is defined by

$$N(P) = \max\{|\alpha_P|^2, |\beta_P|^2\},$$

where α_P and β_P are the eigenvalues of a representative matrix of P.

We call, in general, an infinite product over the shifted values of a zeta function a higher zeta function. The following two higher zeta functions have been appeared in [KW2]:

$$\zeta_{HE}(s) = \prod_{n=0}^{\infty} \zeta(s + 2n), \tag{1.3}$$

(we have denoted ζ_{HE} by ζ_∞ in [KW2]) and

$$z_\Gamma(s) = \prod_{m=1}^{\infty} Z_\Gamma(s + m)^{-1}. \tag{1.4}$$

It is clear that the relation $Z_\Gamma(s) = z_\Gamma(s)/z_\Gamma(s-1)$ holds. In view of the Euler product expression (1.2) of the original Selberg zeta function one easily observes that

$$z_\Gamma(s) = \prod_{m=1}^{\infty} \prod_{P \in \mathrm{Prim}(\Gamma)} (1 - N(P)^{-s-m})^{-m}. \tag{1.5}$$

Beside these examples, our target to concern is the following various types of higher zeta functions

$$\begin{aligned} \zeta_H(s) &= \prod_{n=0}^{\infty} \zeta(s+n) \\ &= \prod_{n=0}^{\infty} \prod_{p:\text{primes}} (1 - p^{-s-n})^{-1}, \end{aligned} \tag{1.6}$$

$$\zeta_{AH}(s) = \prod_{n=1}^{\infty} \zeta(s+n)^{(-1)^n} \tag{1.7}$$

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*) Department of Mathematics, Tokyo Institute of Technology, 2-12-1, O-okayama, Meguro-ku, Tokyo 152-0033.

**) Faculty of Mathematics, Kyushu University, 6-10-1, Hakozaki, Higashi-ku, Fukuoka 812-8581.

Dedicated to Professor Keisaku Kumahara on the occasion of his 60th birthday.

$$= \prod_{n=0}^{\infty} \prod_{p:\text{primes}} (1 - p^{-s-n})^{(-1)^n},$$

and for each $d \in \mathbf{Z}_{>0}$, $\prod_{n=0}^{\infty} \zeta(s+dn+r)$, $\prod_{n=0}^{\infty} \zeta(s+dn+r)^{(-1)^n}$, $(0 \le r < d)$ etc, and further, their analogues for the Dirichlet L-functions. A higher zeta function is sometimes regarded as a generating function. For instance, it is well-known that the higher Riemann zeta function $\zeta_H(s)$ can be considered as a generating function of the order of finite abelian groups (see e.g. [Z]). In fact, let $P_{abel}(n)$ denotes the number of finite abelian groups of order n, that is, $P_{abel}(n) = \#\{G : \text{a finite abelian group; } \#G = n\}$. Then we have

$$\zeta_H(s) := \prod_{n=0}^{\infty} \zeta(s+n) = \prod_{n=0}^{\infty} \prod_{p:\text{primes}} (1 - p^{-s-n})^{-1}$$
$$= \sum_{n=1}^{\infty} \frac{P_{abel}(n)}{n^s}.$$

Motivated by the study of these higher zeta functions we introduce the set

$$\mathrm{Ext}(\varphi) = \{\Phi : \text{meromorphic on } \mathbf{C},\ \Phi(s+1)/\Phi(s) = \varphi(s)\}$$

for a zeta function $\varphi(s)$. We call an element of $\mathrm{Ext}(\varphi)$ a zeta extension of φ. The aim of the paper is to study the following problems and give related questions.

(1) How big is $\mathrm{Ext}(\varphi)$?
(2) Does $\Phi \in \mathrm{Ext}(\varphi)$ have a functional equation when φ has?

We have an answer to the first problem as follows:

Theorem 1. *Let φ be a meromorphic zeta function having an Euler product. Then*

(1) $\mathrm{Ext}(\varphi) \neq \emptyset$.
(2) *If $\Phi_1, \Phi_2 \in \mathrm{Ext}(\varphi)$, then the function $(\Phi_1(s)/\Phi_2(s))$ is periodic, that is, invariant under the translation $s \to s+1$.*

A key point of the proof of (1) is as follows: Put

(1.8)
$$\tilde{\varphi}(s) = \prod_{p \in P} \prod_{n=0}^{\infty} H_p(N(p)^{-s-n}) = \prod_{n=0}^{\infty} \varphi(s+n)^{-1}.$$

Then it is clear that

$$\tilde{\varphi}(s+1) = \tilde{\varphi}(s)\varphi(s). \tag{1.9}$$

This shows that $\tilde{\varphi}(s)$ is meromorphic and hence $\tilde{\varphi} \in \mathrm{Ext}(\varphi)$. The second assertion follows immediately from (1.9). We shall give a detailed proof of this statement when $\varphi(s) = \zeta(s)$ in Theorem 2 in §2.

Example 1. $\zeta_H(s) \in \mathrm{Ext}(\zeta(s)^{-1})$.

Example 2. $z_\Gamma(s) \in \mathrm{Ext}(Z_\Gamma(s+1))$. It is known that $z_\Gamma(s)$ has the functional equation when $\Gamma\backslash H$ is compact as follows: Put

$$\hat{z}_\Gamma(s) = z_\Gamma(s) I(s)^{-(g-1)} \det \cosh \pi \left(\sqrt{\Delta_\Gamma - \frac{1}{4}} - is \right).$$

Here Δ_Γ and g denote respectively the Laplacian and the genus of the Riemann surface $\Gamma\backslash H$, and $I(s)$ is a certain function expressed by the multiple sine functions. Here we note that though the square $I(s)^2$ is meromorphic $I(s)$ itself is not. The functional equation of $z_\Gamma(s)$ is expressed as

$$\hat{z}_\Gamma(s)\hat{z}_\Gamma(-s) = C_\Gamma$$

with some constant C_Γ. See [KW3] for details.

Remark 1. Apart from the usual zeta functions one may ask similar questions for the sine function. Actually, let $\varphi(s)$ be the inverse of the sine function of order 1 (resp. r). If we regard $\varphi(s)$ as a zeta function then the double (resp. order $r+1$) sine function is a zeta extension of φ (see [KKo]). Similar to the problems above one may ask in general the following:

(3) Does $\Phi \in \mathrm{Ext}(\varphi)$ have an addition formula when φ has?

If this (3) has an affirmative answer for the multiple sine functions then it solves the algebraicity problem of the division values of such functions, so we have definitive results for zeta values such as the value of the Riemann zeta function at $3, 5, 7, \ldots$. See [KW1, KOW, KKo].

2. Meromorphic continuations. We give a detailed proof of a meromorphy for various higher Riemann zeta functions. As it will be easily seen, a description of this way of the meromorphic continuation may also provide the exact knowledge of the location of their zeroes and/or poles etc, once we know the explicit information of the one for the original zeta function. We first prove the following theorem.

Theorem 2. *Let $\zeta(s)$ be the Riemann zeta function. For $a > 0$, define*

$$\zeta_H(s,a) = \prod_{n=1}^{\infty} \zeta(s+an),$$
$$\zeta_{AH}(s,a) = \prod_{n=1}^{\infty} \zeta(s+an)^{(-1)^n}.$$

Then $\zeta_H(s,a)$ and $\zeta_{AH}(s,a)$ are meromorophic functions on $\mathbf{C}$.

Proof. By definition we have

$$\zeta_H(s,a) = \prod_{p:\text{primes}} \prod_{n=1}^{\infty} (1-p^{-s-an})^{-1},$$

$$\zeta_{AH}(s,a) = \prod_{p:\text{primes}} \prod_{n=1}^{\infty} (1-p^{-s-an})^{(-1)^{n-1}}.$$

These Euler products converges absolutely in $\mathrm{Re}(s) > 1$ since

$$\sum_p \sum_n p^{-\sigma-an} = \sum_p \frac{p^{-\sigma}}{p^a-1} < +\infty$$

if $\sigma > 1$. Hence we see that $\zeta_H(s,a)$ and $\zeta_{AH}(s,a)$ are holomorphic in $\mathrm{Re}(s) > 1$. Now we have the relations

$$\begin{aligned}\zeta_H(s+a,a) &= \prod_{n=1}^{\infty} \zeta(s+a(n+1)) \\ &= \zeta_H(s,a)\cdot\zeta(s+a)^{-1},\\ \zeta_{AH}(s+a,a) &= \prod_{n=1}^{\infty} \zeta(s+a(n+1))^{(-1)^n} \\ &= \zeta_{AH}(s,a)^{-1}\cdot\zeta(s+a)^{-1}.\end{aligned}$$

Thus we have

$$\zeta_H(s,a) = \zeta_H(s+a,a)\zeta(s+a), \tag{2.1}$$

$$\zeta_{AH}(s,a) = \zeta_{AH}(s+a,a)^{-1}\zeta(s+a)^{-1}. \tag{2.2}$$

From the fact that $\zeta_H(s,a)$ and $\zeta_{AH}(s,a)$ are holomorphic in $\mathrm{Re}(s) > 1$ and the relations (2.1), (2.2), we see that $\zeta_H(s,a)$ and $\zeta_{AH}(s,a)$ are meromorphically continued to the region $\mathrm{Re}(s) > 1-a$. Hence by successive use of the relations (2.1), (2.2), one knows that these are meromorphic in $\mathrm{Re}(s) > 1-am$ for all $m > 1$. This proves $\zeta_H(s,a)$ and $\zeta_{AH}(s,a)$ are meromorphic on $\mathbf{C}$. □

Remark 2. The proof above shows that it is easy to replace ζ by suitable zeta functions, for instance, the Selberg zeta and L-functions, the Hecke L-functions of algebraic number fields.

Remark 3. The proof above also shows that the higher Euler product in Theorem 2 are indeed converges absolutely for $\mathrm{Re}(s) > 1-a$.

It seems, in general, hard to obtain explicitly a desirable functional equation and a gamma factor for higher zeta functions. But, beside this problem, there is a notable example. Actually, it is a result in [CL]. For simplicity, we recall only the case of the Riemann zeta function.

Set

$$W_\infty(s) = 2^{(s-1)(s-2)/4}\pi^{s^2/4} \times \left(\Gamma\left(\frac{s}{2}\right)^{-1}\Gamma_2(s)\right)^{-\frac{1}{2}}\zeta_H(s+1), \tag{2.3}$$

and

$$\Lambda_\infty(s) = W_\infty(s)W_\infty(-s)\frac{\sin^2\pi s}{\pi^2 C_\infty^2}, \tag{2.4}$$

where C_∞ denotes a constant defined by

$$C_\infty = \prod_{j\ge 2}\zeta(j).$$

Then $\Lambda_\infty(s)$ has the following properties:

(1) $\Lambda_\infty(s)$ is an entire function of order 2.
(2) $\Lambda_\infty(s+1) = \Lambda_\infty(s)$.
(3) $\Lambda_\infty(-s) = \Lambda_\infty(s)$.

The periodicity (2) can be deduced from the following functional equation:

$$\hat\zeta(1-s) = \hat\zeta(s), \tag{2.5}$$

where we put $\hat\zeta(s) = \pi^{-(s/2)}\Gamma(s/2)\zeta(s)$.

Theorem 3. *Let $Z(s)$ be an Euler product which is absolutely convergent in $\mathrm{Re}(s) > 1$. Assume that $Z(s)$ has a meromorphic continuation to $\mathbf{C}$ with the functional equation*

$$Z(1-s) = Z(s)\gamma(s)$$

for a suitable factor $\gamma(s)$. Define

$$\Phi(s) = \prod_{n=0}^{\infty} Z(s+n)$$

and put

$$\Psi(s) = \Phi(s)\Phi(2-s).$$

Then

(1) *$\Phi(s)$ is meromorphic on* $\mathbf{C}$,
(2) *$\Psi(s+1) = \Psi(s)\gamma(s)$,*
(3) *$\Psi(-s+1) = \Psi(s+1)$.*

Proof. The proof of (1) is quite similar to the one developed in Theorem 2 using the relation $\Phi(s+1) = \Phi(s)Z(s)^{-1}$. As to the assertion (2), we observe

$$\begin{aligned}\Psi(s+1) &= \Phi(s+1)\Phi(1-s)\\ &= (\Phi(s)Z(s)^{-1})\cdot(\Phi(2-s)Z(1-s))\\ &= \Phi(s)\Phi(2-s)\cdot Z(s)^{-1}Z(1-s)\\ &= \Phi(s)\gamma(s).\end{aligned}$$

The last assertion (3) is clear from the very definition. This completes the proof of the theorem. □

If $Z(s)$ has a complete form $\hat{\zeta}(s)$ like $\zeta(s)$ has (2.5) above, that is $\hat{\zeta}(1-s) = \hat{\zeta}(s)$, then $\Psi(s)$ does have it as exactly the same form.

Corollary 4. *Retain the notation in Theorem 3. Suppose further that the function $\gamma(s)$ is of the form $\gamma(s) = \Gamma_Z(s)/\Gamma_Z(1-s)$ for some function $\Gamma_Z(s)$. Put $\hat{\Psi}(s) = \Psi(s)\Gamma_Z(s)$. Then $\hat{\Psi}(s)$ satisfies $\hat{\Psi}(1-s) = \hat{\Psi}(s)$.*

Proof. Note first that $\Psi(2-s) = \Phi(2-s)\Phi(s) = \Psi(s)$, whence $\Psi(1-s) = \Psi(1+s)$. Since by (2), (3) of Theorem 2, we see that

$$\Psi(1-s) = \Psi(s+1) = \Psi(s)\frac{\Gamma_Z(s)}{\Gamma_Z(1-s)}.$$

Hence the assertion follows immediately. □

Remark 4. The functional equation for $\Phi(s)$ would be of the form

$$\Phi(s)\Phi(2-s) = \text{“}\gamma\text{-factors”}.$$

As to the case $Z(s) = Z_\Gamma(s)$, the Selberg zeta function for a compact Riemann surface, this indeed holds for the higher Selberg zeta function $z_\Gamma(s)$. In fact, the γ-factor consists of a certain determinant of the Laplacian on the Riemann surface and the multiple sine functions. See [KW3] for details.

Remark 5. It would be also interesting to study the n-times iteration $\mathrm{Ext}^n(\varphi) = \mathrm{Ext}\circ\mathrm{Ext}\circ\cdot s\circ\mathrm{Ext}(\varphi)$ and the limit set $\lim_{n\to\infty}\mathrm{Ext}^n(\varphi)$. In particular, does the zeta extension of an element in $\mathrm{Ext}^n(\varphi)$ have a non-trivial functional equation when the element of $\mathrm{Ext}^n(\varphi)$ has? To be explicit, let us explain the situation by using the Selberg zeta function. Recall the Ruell type (Selberg's) zeta function:

$$\zeta_\Gamma(s) = \prod_{P\in\mathrm{Prim}(\Gamma)} (1 - N(P)^{-s})^{-1}.$$

The functional equation of $\zeta_\Gamma(s)$ is known to be given by $\zeta_\Gamma(s)\zeta_\Gamma(-s) = (2\sin\pi s)^{2(2g-2)}$. In our picture, it is clear that

$$\begin{aligned}&\zeta_\Gamma(s) \longrightarrow Z_\Gamma(s) \in \mathrm{Ext}(\zeta_\Gamma(s)) \longrightarrow \\ &z_\Gamma(s-1) \in \mathrm{Ext}(Z_\Gamma(s)) = \mathrm{Ext}^2(\zeta_\Gamma(s)) \longrightarrow \cdots,\end{aligned}$$

and correspondingly the figure of each functional equation varies as

$$\begin{aligned}\zeta_\Gamma(s)\zeta_\Gamma(-s) = f_0(s) &\longrightarrow Z_\Gamma(1-s) = Z_\Gamma(s)f_1(s) \\ &\longrightarrow z_\Gamma(s)z_\Gamma(-s) = f_2(s) \longrightarrow \cdots,\end{aligned}$$

where $f_i(s)$ is an appropriate product of γ-factor. We call these functional equations “non-trivial” which are different from what we established in Theorem 3. What is a next functional equation if any? Moreover, what is the number $N(\varphi)$ defined by

$$\begin{aligned}N(\varphi) = \sup_n\{\exists f\in\mathrm{Ext}^k(\varphi);\ &f \text{ has a non-trivial} \\ &\text{functional equation for } \forall k\le n\}\,?\end{aligned}$$

The above shows $N(\zeta_\Gamma) \ge 2$.

Remark 6. It is known that the function $\Lambda_\infty(s)$ above has the following Weierstrass product representation.

$$\Lambda_\infty(s) = \prod_{\mathrm{Im}\,\rho>0}\left(1 - \frac{\sin^2\pi s}{\sin^2\pi\rho}\right).$$

Here the product is over the non-trivial zeroes of $\zeta(s)$ with positive imaginary part. In particular, $\Lambda_\infty(n) = 1$ for $n\in\mathbf{Z}$. If there is a non-trivial functional equation of $\zeta_H(s)$ it should be of the form $W_\infty(s-1)W_\infty(-s+1) = f_H(s)$ for some $f_H(s)$. Hence one sees that $\Lambda_\infty(s) = f_H(s)\sin^2\pi s/\pi^2C_\infty^2$ by (2.4). In particular, $f_H(s)$ must be periodic. It would be interesting if one compares the coefficients of the both sides of this quation relative to the expansion for $\sin^2\pi s$. We will deal with this problem in the future.

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References

[CL] Cohen, H., and Lenstra, H. W.: Heuristics on class groups of number fields. Lecture Notes in Math., no. 1068, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, pp. 33–62 (1984).

[KKo] Kurokawa, N., and Koyama, S.: Multiple sine functions. Forum. Math. (To appear).

[KOW] Kurokawa, N., Ochiai, H., and Wakayama, M.: Zetas and multiple trigonometry. J. Ramanujan Math. Soc., **17**, 101–113 (2002).

[KW1] Kurokawa, N., and Wakayama, M.: On $\zeta(3)$. J. Ramanujan Math. Soc., **16**, 205–214 (2001).

[KW2] Kurokawa, N., and Wakayama, M.: A comparison between the sum over Selberg's zeroes and Riemann's zeroes. (2002). (Preprint).

[KW3] Kurokawa, N., and Wakayama, M.: Higher Selberg zeta functions. (2002). (Preprint).

[Z] Zagier, D.: Zetafunktionen und quadratische körper. Springer-Verlag, Berlin-Heidelberg (1981).