The primitive derivation and freeness of multi-Coxeter arrangements

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Abstract: We will prove the freeness of multi-Coxeter arrangements by constructing a basis of the module of vector fields which contact to each reflecting hyperplanes with some multiplicities using K. Saito's theory of primitive derivation.

Key words: Hodge filtration; finite reflection group; Coxeter arrangement; adjoint quotient.

1. Introduction. Let V be a Euclidean space over $\mathbf{R}$ with finite dimension ℓ and inner product I. Let $W \subset \mathrm{O}(V, I)$ be a finite irreducible reflection group and $\mathcal{A}$ the corresponding Coxeter arrangement i.e. the collection of all reflecting hyperplanes of W. For each $H \in \mathcal{A}$, we fix a defining equation $\alpha_H \in V^*$ of H.

In [5], H. Terao constructed a free basis of $\mathbf{R}[V]$-module

$$(1) \qquad \mathrm{D}^m(\mathcal{A}) := \{\delta \in \mathrm{Der}_V \mid \delta\alpha_H \in (\alpha_H^m),\ \forall H \in \mathcal{A}\},$$

($m \in \mathbf{Z}_{\geq 0}$). The purpose of this paper is to construct a basis by a simpler way using Saito's result and give a generalization.

For given multiplicity $\tilde{m} : \mathcal{A} \to \mathbf{Z}_{\geq 0}$, we say that the multi-Coxeter arrangement $\mathcal{A}^{(\tilde{m})}$ is free if the module

$$(2) \qquad \mathrm{D}(\mathcal{A}^{(\tilde{m})}) := \{\delta \in \mathrm{Der}_V \mid \delta\alpha_H \in (\alpha_H^{\tilde{m}(H)}),\ \forall H \in \mathcal{A}\}$$

is a free $\mathbf{R}[V]$-module [8]. Then our main result is

Theorem 1. *Let $\tilde{m}$ be a multiplicity satisfying $\tilde{m}(H) \in \{0, 1\}$ for all $H \in \mathcal{A}$. Suppose the multi-Coxeter arrangement $\mathcal{A}^{(\tilde{m})}$ is free, then $\mathcal{A}^{(\tilde{m}+2k)}$ ($k \in \mathbf{Z}_{\geq 0}$) is also free, where the new multiplicity $\tilde{m} + 2k$ take value $\tilde{m}(H) + 2k$ at $H \in \mathcal{A}$.*

We construct a basis in Theorem 7.

We note that $\mathcal{A}^{(\tilde{m})}$ is not necesarily free for $\tilde{m} : \mathcal{A} \to \{0, 1\}$. If we apply Theorem 1 for $\tilde{m}(H) \equiv 0$ or $\tilde{m}(H) \equiv 1$, we obtain the freeness of $\mathrm{D}^{2k}(\mathcal{A})$ or $\mathrm{D}^{2k+1}(\mathcal{A})$. Terao's basis is expected to coincide with that of ours.

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The original motivation to study the module $\mathrm{D}(\mathcal{A}^{(\tilde{m})})$ came from the study of structures of the relative de Rham cohomology $\mathrm{H}^*(\Omega^\bullet_{\mathfrak{g}/S})$ of the adjoint quotient map $\chi : \mathfrak{g} \to S := \mathfrak{g}//\mathrm{ad}(G)$ of a simple Lie algebra $\mathfrak{g}$. In the case of ADE type Lie algebras, an isomorphism as $\mathbf{C}[S] (= \mathbf{C}[\mathfrak{g}]^G = \mathbf{C}[\mathfrak{h}]^W)$-modules (where $\mathfrak{h}$ is a Cartan subalgebra)

$$\mathrm{H}^2(\Omega^\bullet_{\mathfrak{g}/S}) \cong \mathrm{D}^5(\mathcal{A})^W$$

is obtained [7].

But for $BCFG$ type Lie algebras, because the W action on $\mathcal{A}$ is not transitive, $\mathrm{H}^2(\Omega^\bullet_{\mathfrak{g}/S})$ is expected to be isomorphic to the module $\mathrm{D}(\mathcal{A}^{(\tilde{m})})^W$ with a suitable multiplicity $\tilde{m} : \mathcal{A} \to \mathbf{Z}_{\geq 0}$ which is not constant.

2. K. Saito's results on primitive derivation. In this section, we fix notations and recall some results.

Let $x_1, \dots, x_\ell \in V^*$ be a basis of V^* and $P_1, P_2, \cdots, P_\ell \in \mathbf{R}[V]^W$ be the homogeneous generators of W-invariant polynomials on V such that $\mathbf{R}[V]^W = \mathbf{R}[P_1, P_2, \cdots, P_\ell]$ with

$$\deg P_1 \leq \deg P_2 \leq \cdots \leq \deg P_\ell =: h.$$

Then it is classically known [1] that

$$(3) \qquad |\mathcal{A}| = \frac{h\ell}{2}$$

and

$$(4) \qquad \deg P_{\ell-1} < h.$$

It follows from (4) that the rational vector field (with pole along $\bigcup_{H\in\mathcal{A}} H$) $D := (\partial/\partial P_\ell)$ on V is uniquely determined up to non-zero constant factor independently on the generators $P_1, \dots, P_\ell$. We call D the primitive vector field. If we fix generators $P_1, \dots, P_\ell$, then $(\partial/\partial P_1), \dots, (\partial/\partial P_{\ell-1})$ are able to be considered as rational vector fields on V. Since the Jacobian is

$$Q := \prod_{H \in \mathcal{A}} \alpha_H \doteq \frac{\partial(P_1, \dots, P_\ell)}{\partial(x_1, \dots, x_\ell)},$$

D is symbolically expressed as

$$D \doteq \frac{1}{Q} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix}.$$

Next we define an affine connection $\nabla : \mathrm{Der}_V \times \mathrm{Der}_V \to \mathrm{Der}_V$.

Definition 2. For given $\delta_1, \delta_2 \in \mathrm{Der}_V$ with $\delta_2 = \sum_{i=1}^{\ell} f_i (\partial/\partial x_i)$,

$$\nabla_{\delta_1} \delta_2 := \sum_{i=1}^{\ell} (\delta_1 f_i) \frac{\partial}{\partial x_i}.$$

The connection ∇ can be also characterized by the formula:

$$(5) \quad (\nabla_{\delta_1} \delta_2)\alpha = \delta_1(\delta_2 \alpha), \ \forall \text{ linear function } \alpha \in V^*.$$

This formula plays an important role in our computations.

The derivation ∇_D by the primitive vector field is particularly important. Define $\mathbf{R}[V]^{W,\tau} := \{f \in \mathbf{R}[V]^W \mid Df = 0\} = \mathbf{R}[P_1, \dots, P_{\ell-1}]$. Then ∇_D is an $\mathbf{R}[V]^{W,\tau}$-homomorphism. The following decomposition of $\mathrm{Der}_V^W = \mathrm{D}^1(\mathcal{A})^W$ has been obtained in [2, 3].

Theorem 3. *Let $n \geq 1$, define*

$$\mathcal{G}_n := \left\{ \delta \in \mathrm{Der}_V^W \;\middle|\; (\nabla_D)^n \delta \in \sum_{i=1}^{\ell} \mathbf{R}[V]^{W,\tau} \frac{\partial}{\partial P_i} \right\},$$

then for every $n \geq 0$, ∇_D induces an $\mathbf{R}[V]^{W,\tau}$-isomorphism $\mathcal{G}_{n+1} \tilde{\to} \mathcal{G}_n$ and

$$\mathrm{D}^1(\mathcal{A})^W = \bigoplus_{n \geq 1} \mathcal{G}_n.$$

If we define $\mathcal{H}^k := \bigoplus_{n \geq k} \mathcal{G}_n$, then it becomes a rank ℓ free $\mathbf{R}[V]^W$-submodule of Der_V^W, which is called the Hodge filtration.

In particular, $\nabla_D : \mathcal{H}^2 \tilde{\to} \mathcal{H}^1 = \mathrm{D}^1(\mathcal{A})^W$ is an $\mathbf{R}[V]^{W,\tau}$-isomorphism. All we need in the sequel is the existence of an injection $\nabla_D^{-1} : \mathrm{Der}_V^W \to \mathrm{Der}_V^W$.

3. Construction of a basis.

We construct a basis of $\mathrm{D}(\mathcal{A}^{(2k+\tilde{m})})$. The following is a key lemma which connects two filtrations, the Hodge filtration and the contact-order filtration.

Lemma 4. *Let $\delta', \delta \in \mathrm{Der}_V$ be vector fields on V and assume $\nabla_D \delta' = \delta$. Then for any $H \in \mathcal{A}$, $\delta \alpha_H$ is divisible by α_H^m if and only if $\delta' \alpha_H$ is divisible by α_H^{m+2}.*

Proof. Suppose $\delta' \alpha = \alpha^{m'} f$ (where $\alpha = \alpha_H$). Then from (5),

$$(6) \quad (\nabla_D \delta')\alpha = D(\delta' \alpha)$$
$$= \frac{1}{Q} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} & \frac{\partial}{\partial x_1}(\alpha^{m'} f) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_\ell} & \frac{\partial}{\partial x_\ell}(\alpha^{m'} f) \end{pmatrix}.$$

Thus $\delta\alpha$ is divisible by $\alpha^{m'-2}$. Further, assume f is not divisible by α, let us show that $\delta\alpha$ is not divisible by $\alpha^{m'-1}$. Take a coordinate system $x_1, \dots, x_{\ell-1}, x_\ell$ such that $x_\ell = \alpha$, then it suffices to show that

$$\det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial P_1}{\partial x_{\ell-1}} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_{\ell-1}} \end{pmatrix} \text{ is not divisible by } \alpha.$$

After taking $\otimes \mathbf{C}$ and restricting to $H_{\mathbf{C}} := H \otimes \mathbf{C}$, determinant above can be interpreted as the Jacobian of the composed mapping

$$\begin{array}{cccc} \phi : & H_{\mathbf{C}} & \to & \mathrm{Spec}\mathbf{C}[V]^{W,\tau} \\ & (x_1, \dots, x_{\ell-1}, 0) & \mapsto & (P_1, \dots, P_{\ell-1}). \end{array}$$

On the other hand, since the set

$$\{x \in V \otimes \mathbf{C} \mid P_1(x) = \cdots = P_{\ell-1}(x) = 0\}$$

is a union of some eigenspaces of Coxeter transformations in W, which are regular, that is, they intersect with $H_{\mathbf{C}}$ only at $0 \in H_{\mathbf{C}}$ [2, 3]. Hence $\phi^{-1}(0) = \{0\} \subset H_{\mathbf{C}}$, and the Jacobian of ϕ cannot be identically zero. □

Remark 5. The precise expression of the Jacobian of ϕ is obtained in [4]. It is equal to the reduced defining equation of the union of hyperplanes $\bigcup_{H' \in \mathcal{A} \setminus \{H\}} (H \cap H')$, on H.

Because of Theorem 3 the operator ∇_D^{-1} is well defined on $\mathrm{Der}_V^W = \mathrm{D}^1(\mathcal{A})^W$, we have

Lemma 6. *Let $\delta \in \mathrm{Der}_V^W$ be a W-invariant vector field on V. Then for any $H \in \mathcal{A}$, $\delta \alpha_H$ is* *divisible by* α_H^m *if and only if* $(\nabla_D^{-1}\delta)\alpha_H$ *is divisible by* α_H^{m+2}.

By induction with $\mathcal{H}^1 = \mathrm{D}^1(\mathcal{A})^W$, Lemma 6 indicates

$$(7) \qquad \mathcal{H}^k = \nabla_D^{-k+1}\mathrm{D}^1(\mathcal{A})^W \subset \mathrm{D}^{2k+1}(\mathcal{A})^W.$$

The converse is also true, which will be proved in §4.

We denote by $E := \sum_{i=1}^{\ell} x_i(\partial/\partial x_i)$ the Euler vector field. Note that E is contained in $\mathrm{D}^1(\mathcal{A})^W$, $\nabla_\delta E = \delta$ and $\nabla_E \delta = (\deg\delta)\delta$ for any homogeneous vector field $\delta \in \mathrm{Der}_V$. By Theorem 3, we have a "universal" vector field $\nabla_D^{-k}E$.

As in §1, let $\tilde{m} : \mathcal{A} \to \{0,1\}$ be a multiplicity and assume that $\delta_1, \delta_2, \ldots, \delta_\ell \in \mathrm{D}(\mathcal{A}^{(\tilde{m})})$ be a free basis of the multiarrangement $\mathcal{A}^{(\tilde{m})}$.

Theorem 7. *Under the above hypothesis,* $\nabla_{\delta_1}\nabla_D^{-k}E, \ldots, \nabla_{\delta_\ell}\nabla_D^{-k}E$ *form a free basis of* $\mathrm{D}(\mathcal{A}^{(\tilde{m}+2k)})$.

Proof. Let $\delta \in \mathrm{D}(\mathcal{A}^{(\tilde{m})})$, we first prove $\nabla_\delta\nabla_D^{-k}E \in \mathrm{D}(\mathcal{A}^{(\tilde{m}+2k)})$. From (7), $\nabla_D^{-k}E \in \mathrm{D}^{2k+1}(\mathcal{A})$, we may assume

$$(8) \qquad (\nabla_D^{-k}E)\alpha = \alpha^{2k+1}f$$

for $\alpha = \alpha_H$, $(H \in \mathcal{A})$. Applying δ to the both sides of (8), we have

$$(9) \qquad (\nabla_\delta\nabla_D^{-k}E)\alpha = \alpha^{2k}((2K+1)(\delta\alpha)f + \alpha(\delta f)).$$

Since $\delta\alpha$ is divisible by α with multiplicity $\tilde{m}(H) \le 1$, hence $(\nabla_\delta\nabla_D^{-k}E)\alpha$ is divisible by $\alpha^{\tilde{m}(H)+2k}$. □

Here we recall G. Ziegler's criterion on freeness of multiarrangements.

Theorem 8 [8]**.** *Let* $\tilde{m} : \mathcal{A} \to \mathbf{Z}_{\ge 0}$ *be a multiplicity and* $\delta_1, \ldots, \delta_\ell \in \mathrm{D}(\mathcal{A}^{(\tilde{m})})$ *be homogeneous and linearly independent over* $\mathbf{C}[V]$*. Then* $\mathcal{A}^{(\tilde{m})}$ *is free with basis* $\delta_1, \ldots, \delta_\ell$ *if and only if*

$$\sum_{i=1}^{\ell} \deg\delta_i = \sum_{H\in\mathcal{A}} \tilde{m}(H).$$

We compute the degrees of $\nabla_{\delta_1}\nabla_D^{-k}E, \ldots, \nabla_{\delta_\ell}\nabla_D^{-k}E$,

$$(10) \qquad \begin{aligned} \sum_{i=1}^{\ell} \deg(\nabla_{\delta_i}\nabla_D^{-k}E) &= \sum_{i=1}^{\ell}(kh + \deg\delta_i) \\ &= kh\ell + \sum_{i=1}^{\ell}\deg\delta_i, \end{aligned}$$

where $h = \deg P_\ell$ is the Coxeter number. On the other hand, the sum of multiplicities is

$$(11) \qquad \sum_{H\in\mathcal{A}}(\tilde{m}(H) + 2k) = 2k|\mathcal{A}| + \sum_{H\in\mathcal{A}}\tilde{m}(H).$$

The assumption implies $\sum_{H\in\mathcal{A}}\tilde{m}(H) = \sum_{i=1}^{\ell}\deg\delta_i$ and because of (3), we conclude that (10) coincides with (11).

4. Some conclusions.

Lemma 9. $\nabla_{(\partial/\partial P_i)}\mathrm{D}^{2k+1}(\mathcal{A})^W \subset \mathrm{D}^{2k-1}(\mathcal{A})^W$ $(k > 0)$.

Proof. We only prove for $i = \ell$, remaining cases can be proved similarly. It is sufficient to show that $(\nabla_D\delta)\alpha_{H_0}$ has no poles for any $\delta \in \mathrm{D}^{2k+1}(\mathcal{A})^W$ and $H_0 \in \mathcal{A}$. By (6), $QD\delta\alpha_{H_0}$ can be divided by α_{H_0}, so all we have to show is that $QD\delta\alpha_{H_0}$ is divisible by $\beta := \alpha_{H'}$ for all $H' \in \mathcal{A}\backslash\{H_0\}$. We denote by $s_\beta \in W$ the reflection with respect to the hyperplane $H' \subset V$, then $s_\beta(\alpha)$ is expressed in the form $s_\beta(\alpha) = \alpha + 2c\beta$ for some $c \in \mathbf{R}$. Apply s_β to the function $QD\delta\alpha$, since D and δ are W-invariant, and $s_\beta(Q) = -Q$,

$$s_\beta(QD\delta\alpha) = -QD\delta\alpha - 2cQD\delta\beta.$$

By using the equation $s_\beta(QD\delta\beta) = QD\delta\beta$, we have

$$s_\beta(QD\delta\alpha + cQD\delta\beta) = -(QD\delta\alpha + cQD\delta\beta).$$

So $QD\delta\alpha + cQD\delta\beta$ is divisible by β, but from the first half of this proof, $cQD\delta\beta$ is divisible by β, and the other term $QD\delta\alpha$ is also divisible by β. □

As a consequence of induction, we have

Corollary 10 [6]**.** $\mathcal{H}^k = \mathrm{D}^{2k+1}(\mathcal{A})^W$.

Finally, we apply Theorem 1 to $\tilde{m} \equiv 0$ or $\tilde{m} \equiv 1$, since both $\mathrm{D}^0(\mathcal{A}) = \mathrm{Der}_V$ and $\mathrm{D}^1(\mathcal{A})$ are free, we obtain

Corollary 11 [5]**.** $\mathrm{D}^m(\mathcal{A})$ *is free for all* $m \ge 0$.

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