On some isotropic submanifolds in spheres

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Abstract: We give examples of isotropic submanifolds in spheres, which are counter-examples to the result of [S].

Key words: Isotropic; constant mean curvature; sphere.

1. Introduction. Let $f : M^n \to \widetilde{M}^{n+p}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$. We recall the notion of isotropic immersions ([O]): Let σ be the second fundamental form of M^n in $\widetilde{M}^{n+p}$. Then the immersion f is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2$ does not depend on the choice of $X(\neq 0) \in T_xM$. If the immersion is isotropic at every point, then there exists a function λ on M defined by $x \mapsto \|\sigma(X,X)\|/\|X\|^2$ and the immersion f is said to be λ-isotropic or, simply, isotropic. If the function λ is constant on M^n, we call (M^n, f) a constant isotropic submanifold.

Note that a totally umbilic immersion is isotropic, but not *vice versa*. There are many examples of isotropic submanifolds which are not totally umbilic in standard spheres.

The purpose of this paper is to construct examples of isotropic immersions of spheres into spheres satisfying the following theorem:

Theorem. *There exist many compact connected isotropic submanifolds M^n's of an $(n+p)$-dimensional sphere $S^{n+p}(\tilde{c})$ of curvature $\tilde{c}$ satisfying the following three conditions*:

(i) *M^n has constant mean curvature $H(:= \|\mathfrak{h}\|)$, where $\|\mathfrak{h}\|$ is the length of the mean curvature vector $\mathfrak{h}$ of M^n in $S^{n+p}(\tilde{c})$.*
(ii) *The sectional curvatures K of M^n are greater than or equal to $(H^2+\tilde{c})/2$.*
(iii) *M^n is not totally umbilic in $S^{n+p}(\tilde{c})$.*

This provides us with many counter-examples to the result of Y. B. Shen ([S]). He showed that if a compact connected isotropic submanifold M^n of an $(n+p)$-dimensional sphere $S^{n+p}(\tilde{c})$ of curvature $\tilde{c}$ satisfies the conditions (i) and (ii) in our Theorem, then the submanifold M^n is totally umbilic in $S^{n+p}(\tilde{c})$. However unfortunately by virtue of our Theorem the author claims that his proof in [S] has an error.

2. A construction of isotropic immersions. We denote by $S^N(c)$ an N-dimensional sphere of curvature c. Let M^n be an n-dimensional compact isotropy-irreducible Riemannian homogeneous space. We choose two minimal isotropic immersions of M^n into spheres, say χ_1 and χ_2. We set $\chi_1 : M^n \to S^{N_1}(c_1)$ and $\chi_2 : M^n \to S^{N_2}(c_2)$. Here, χ_1 (resp. χ_2) is a minimal λ_1-(resp. λ_2-) isotropic immersion with respect to some eigenvalue μ_1 (resp. μ_2) of the Laplacian of M^n. Suppose that $\mu_1 \neq \mu_2$. It is well-known that $c_1 = \mu_1/n$ and $c_2 = \mu_2/n$ (for details, see [T]). By using these two minimal isotropic immersions χ_1 and χ_2, we construct the following examples of isotropic immersions of M^n into spheres.

Example. For each $t \in (0, \pi/2)$ the isometric immersion $f_t : M^n \to S^N(\tilde{c})$ is given by

$$(2.1)\quad f_t :\ M^n \overset{(\chi_1,\chi_2)}{\longrightarrow} S^{N_1}\Big(\frac{c_1}{\cos^2 t}\Big) \times S^{N_2}\Big(\frac{c_2}{\sin^2 t}\Big) \longrightarrow S^N(\tilde{c}),$$

where $N = N_1 + N_2 + 1$ and $\cos^2 t/c_1 + \sin^2 t/c_2 = 1/\tilde{c}$. Here the differential map $(\chi_1,\chi_2)_*$ of (χ_1,χ_2) is defined by

$$(2.2)\quad (\chi_1,\chi_2)_*X := (\cos t\cdot(\chi_1)_*X, \sin t\cdot(\chi_2)_*X) \quad \text{for each } X \in TM^n.$$

Needless to say, the $S^{N_1}(c_1/\cos^2 t) \times S^{N_2}(c_2/\sin^2 t)$ is imbedded into $S^N(\tilde{c})$ as a Clifford hypersurface.

Our aim here is to clarify geometric properties of the immersion f_t given by (2.1).

Proposition. *For each $t \in (0,\pi/2)$ the isometric immersion $f_t : M^n \to S^N(\tilde{c})$ given by (2.1) has the following geometric properties*:

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(1) *f_t has nonzero constant mean curvature.*
(2) *f_t is isotropic.*
(3) *f_t is pseudo umbilic but not totally umbilic.*
(4) *The mean curvature vector $\mathfrak{h}_t$ of f_t is not parallel.*

Proof. We shall compute the second fundamental form σ_t of f_t. First we consider the second fundamental form of $(\chi_1, \chi_2) : M^n \to S^{N_1}(c_1/\cos^2 t) \times S^{N_2}(c_2/\sin^2 t)$. We denote by σ_1 (resp. σ_2) the second fundamental form of χ_1 (resp. χ_2). Then it follows from (2.2) that the second fundamental form of (χ_1, χ_2) is given by $(\cos^2 t\cdot\sigma_1, \sin^2 t\cdot\sigma_2)$, so that $(\chi_1, \chi_2) : M^n \to S^{N_1}(c_1/\cos^2 t) \times S^{N_2}(c_2/\sin^2 t)$ is a minimal isotropic immersion. Next, we study the second fundamental form of the Clifford hypersurface given by (2.1). For simplicity we put $\widetilde{c_1} = c_1/\cos^2 t$ and $\widetilde{c_2} = c_2/\sin^2 t$. We choose a unit normal vector field ξ of $S^{N_1}(\widetilde{c_1}) \times S^{N_2}(\widetilde{c_2})$ in $S^N(\tilde{c})$. Let A be the shape operator of this hypersurface in $S^N(\tilde{c})$ with respect to ξ. Then it is well-known that A is expressed as follows:

$$A = \Big(\frac{\widetilde{c_1}}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}} I_{N_1}\Big) \oplus \Big(\frac{-\widetilde{c_2}}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}} I_{N_2}\Big), \tag{2.3}$$

where I_k is an identity matrix of degree k. We need to calculate $A|_{TM^n}$ which is the restriction of the shape operator A on the tangent bundle TM^n. By easy computation from (2.2) and (2.3) we know that

$$A|_{TM^n} = \frac{\cos^2 t\cdot\widetilde{c_1} - \sin^2 t\cdot\widetilde{c_2}}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}} I_n.$$

Therefore the second fundamental form σ_t is expressed as follows:

$$\sigma_t(X,Y) = (\cos^2 t\cdot\sigma_1(X,Y), \sin^2 t\cdot\sigma_2(X,Y)) + \frac{\cos^2 t\cdot\widetilde{c_1} - \sin^2 t\cdot\widetilde{c_2}}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}}\langle X,Y\rangle\xi \tag{2.4}$$

for any vector fields X, Y on M^n, where $\widetilde{c_1} = c_1/\cos^2 t$ and $\widetilde{c_2} = c_2/\sin^2 t$. Hence the mean curvature vector $\mathfrak{h}_t := (1/n)\,\mathrm{trace}\,\sigma_t$ is given by

$$\mathfrak{h}_t = \frac{\cos^2 t\cdot\widetilde{c_1} - \sin^2 t\cdot\widetilde{c_2}}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}}\xi. \tag{2.5}$$

Then the mean curvature H_t is

$$H_t := \|\mathfrak{h}_t\| = \frac{|\cos^2 t\cdot\widetilde{c_1} - \sin^2 t\cdot\widetilde{c_2}|}{\sqrt{\widetilde{c_1}+\widetilde{c_2}}}. \tag{2.6}$$

We are now in a position to prove geometric properties of f_t.

(1) Equation (2.6) guarantees the constancy of H_t. Here note that $H_t \neq 0$ for each $t \in (0, \pi/2)$. In fact, suppose that $H_{t_0} = 0$ for some t_0. Then by the well-known result of Takahashi [T] the immersion $f_{t_o} : M^n \to S^N(\tilde{c})$ is represented by eigenfunctions of some eigenvalue μ of the Laplacian on M^n, which is a contradiction (see the definitions of χ_1 and χ_2).

(2) It follows from (2.4) that the immersion f_t is (λ_t-)isotropic. λ_t is given by

$$\lambda_t = \sqrt{\cos^4 t\cdot\lambda_1^2 + \sin^4 t\cdot\lambda_2^2 + \frac{(\cos^2 t\cdot\widetilde{c_1} - \sin^2 t\cdot\widetilde{c_2})^2}{\widetilde{c_1}+\widetilde{c_2}}}.$$

(3) From (2.4) and (2.5) we can see that $\langle\sigma_t(X,Y), \mathfrak{h}_t\rangle = \langle X,Y\rangle\|\mathfrak{h}_t\|^2$ for each $X, Y \in TM^n$, so that f_t is pseudo umbilic, but of course it is not totally umbilic.

(4) We denote by D the normal connection of M^n in $S^N(\tilde{c})$. We choose a local field of orthonormal frames $e_1, \dots, e_n$ on M^n. Then by direct computation we find that for each $i \in \{1, \dots, n\}$ the normal vector $D_{e_i}\mathfrak{h}$ contains some nonzero scalar multiple of the vector $(\sin t\cdot(\chi_1)_* e_i, -\cos t\cdot(\chi_2)_* e_i)$ which is normal to M^n. □

3. Proof of Theorem. In Example (2.1) first of all we set $M^n = S^n(n/(2(n+1)))$, and put χ_1 and χ_2 as follows: Let $\chi_1 : S^n(n/(2(n+1))) \to S^{n+(n(n+1)/2)-1}(1)$ be the second standard minimal immersion and

$$\chi_2 : S^n\Big(\frac{n}{2(n+1)}\Big) \to S^n\Big(\frac{n}{2(n+1)}\Big)$$

be the identity mapping. Then we see that $\lambda_1 = \sqrt{(n-1)/(n+1)}$ and $\lambda_2 = 0$. We particularly put $\cos t = 1/\sqrt{n+1}$, $\sin t = \sqrt{n/(n+1)}$. Hence we obtain the following constant isotropic submanifold $S^n(n/(2(n+1)))$ with constant mean curvature, say H in $S^{n+n(n+3)/2}((n+1)/(2n+3))$.

$$S^n\Big(\frac{n}{2(n+1)}\Big) \xrightarrow{\text{minimal}} S^{n+(n(n+1)/2)-1}(n+1) \times S^n\Big(\frac{1}{2}\Big) \longrightarrow S^{n+n(n+3)/2}\Big(\frac{n+1}{2n+3}\Big).$$

Then from (2.6) we can find that

$$H = \frac{n+2}{(n+1)\sqrt{2(2n+3)}}.$$

This, together with $K = n/2(n+1)$ and $\tilde{c} = (n+1)/(2n+3)$, yields that

$$K - \frac{1}{2}(H^2 + \tilde{c}) = \frac{n^2 - 2}{4(n+1)^2} > 0.$$

Thus we get the conclusion.

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