A theory of genera for cyclic coverings of links

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Abstract: Following the conceptual analogies between knots and primes, 3-manifolds and number fields, we discuss an analogue in knot theory after the model of the arithmetical theory of genera initiated by Gauss. We present an analog for cyclic coverings of links following along the line of Iyanaga-Tamagawa's genus theory for cyclic extentions over the rational number field. We also give examples of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$-coverings of links for which the principal genus theorem does not hold.

Key words: Links; genus and central class coverings; genera of homology classes.

1. We first introduce the notions of genus and central class coverings and genera of homology classes in the context of 3-dimensional topology, following the arithmetical theory due to Iyanaga-Tamagawa ([IT]) and Leopoldt ([L]). For the foundational analogies bridging 3-dimentional topology and algebraic number theory, we refer to [Ma], [W], [RKM], [R1,2,3] and [Mo1,2,3]. This note is a partial and group-theoretic result in the topic raised in the end of §2 of [Mo2].

Let $L = K_1 \cup \cdots \cup K_\mu$ be a tame link consisting of μ knots K_i's in the 3-sphere S^3. Let $p : M \to S^3$ be a finite abelian covering ramified along L with Galois group G. Such a covering M is obtained by completing an unramified covering $Y \to X := S^3 \setminus L$ corresponding to the kernel of a surjective homomorphism $\psi : H_1(X, \mathbf{Z}) \to G$. We assume that M is a rational homology 3-sphere so that the 1st homology group $H_1(M, \mathbf{Z})$ is finite. Let M^a be the maximal abelian (unramified) covering of M with Galois group H being isomorphic to $H_1(M, \mathbf{Z})$ by the Hurewicz theorem. Then, the composite covering $M^a \to S^3$ is a Galois covering with metabelian Galois group Γ. We define M^g and M^z to be the Galois coverings of S^3 corresponding to the subgroups $[\Gamma, \Gamma]$ and $[\Gamma, H]$ of Γ, respectively, by Galois theory of coverings, where $[A, B]$ stands for the group generated by commutators $aba^{-1}b^{-1}$, $a \in A$, $b \in B$. Note that M^g is the maximal abelian covering of S^3 containing M as a subcovering and that M^z is the maximal covering of S^3 so that the Galois group $\mathrm{Gal}(M^a/M^z)$ is contained in the center of Γ. We call M^g and M^z the *genus covering* and *central class covering* of M over S^3 respectively. The subgroup of $H_1(M, \mathbf{Z})$ corresponding to $[\Gamma, \Gamma]$ under the Hurewicz isomorphism $H_1(M, \mathbf{Z}) \simeq H$ is called the *principal genus group* of M over S^3 and denoted by G_M. Note that $[\Gamma, H]$ is given by $I_G H_1(M, \mathbf{Z})$ as a subgroup $H_1(M, \mathbf{Z})$, where I_G denotes the augmentation ideal of the integral group ring $\mathbf{Z}[G]$.

$$\begin{array}{ccl}
M^a & \leftrightarrow & \{e\} \\
\downarrow & & \cap \\
M^z & \leftrightarrow & [\Gamma, H] = I_G H_1(M, \mathbf{Z}) \\
\downarrow & & \cap \\
M^g & \leftrightarrow & [\Gamma, \Gamma] = G_M \\
\downarrow & & \cap \\
M & \leftrightarrow & H = H_1(M, \mathbf{Z}) \\
\downarrow & & \cap \\
S^3 & \leftrightarrow & \Gamma
\end{array}$$

An element of the quotient $H_1(M, \mathbf{Z})/G_M$ is called a *genus* in $H_1(M, \mathbf{Z})$ and the number of genera $g_M := [M^g : M]$ is called the *genus number* of M over S^3.

Based on the analogy between norm residue symbols and linking numbers ([Ma], [Mo1,2], [RKM], [W]), we introduce the following "system of characters" on the homology group $H_1(M, \mathbf{Z})$ to characterize the group of genera. Let e_i (> 1) be the order of the meridian class t_i around K_i in G under $\psi : H_1(X, \mathbf{Z}) \to G$. We then define the homomorphism

$$\Phi : H_1(M, \mathbf{Z}) \longrightarrow \prod_{i=1}^{\mu} \mathbf{Z}/e_i\mathbf{Z}$$

by

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$$\Phi(c) := (\mathrm{lk}(p_*(c), K_i) \bmod e_i)_{1\le i\le\mu}$$

where $\mathrm{lk}(*,*)$ stands for the linking number of homology classes. We find that the well-definedness of the map Φ is shown implicitly in [S2], Lemma 1. In fact, our map Φ is nothing but the homomorphism

$$\begin{aligned} &H_1(M,\mathbf{Z}) = H_1(Y,\mathbf{Z})/(\text{branch relation}) \\ &\to H_1(X,\mathbf{Z})/\langle e_i t_i (1\le i\le\mu)\rangle \end{aligned}$$

induced by $p_* : H_1(Y,\mathbf{Z}) \to H_1(X,\mathbf{Z})$, from which the well-definedness of Φ follows.

Now, the following theorem is regarded as an analog of Iyanaga-Tamagawa's genus theory ([IT]) for cyclic coverings of links.

Theorem. *Notations and assumptions being as above, we assume further that $p : M \to S^3$ is a cyclic covering with Galois group G generated by an element σ of order n. Set $\psi(t_i) = \sigma^{a_i}$ $(0 < a_i < n)$. Then we have*

(1) *$G_M = (\sigma-1)H_1(M,\mathbf{Z})$ and hence $M^g = M^z$, and the genus number equals the number of ambiguous homology classes, $g_M = \{c \in H_1(M,\mathbf{Z}) \mid \sigma(c) = c\}$.*

(2) *The map Φ gives an isomorphism*

$$\begin{aligned} &H_1(M,\mathbf{Z})/(\sigma-1)H_1(M,\mathbf{Z}) \\ &\simeq \left\{ (\xi_i \bmod e_i) \in \prod_{i=1}^{\mu} \mathbf{Z}/e_i\mathbf{Z} \;\middle|\; \sum_{i=1}^{\mu} a_i\xi_i \equiv 0 \bmod n \right\}. \end{aligned}$$

In particular, we have

$$g_M = \frac{\prod_{i=1}^{\mu} e_i}{n}.$$

For the case of $n = 2$, we easily see that $(\sigma-1)H_1(M,\mathbf{Z}) = 2H_1(M,\mathbf{Z})$ and obtain the Gauss genus theory ([G]) for links.

Corollary. *Assume further that $n = 2$. Then we have*

$$H_1(M,\mathbf{Z})/2H_1(M,\mathbf{Z}) \simeq (\mathbf{Z}/2\mathbf{Z})^{\mu-1}.$$

2. Here is a group-theoretic proof of our theorem. Firstly, we see the equality $[\Gamma,\Gamma] = I_G H = (\sigma-1)H$ when G is cyclic, since we have the exact sequence $0 \to G^* \to (\Gamma/[\Gamma,\Gamma])^* \to (H/I_GH)^* \to 0$ $(A^* := \mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z}))$ by the Hochschild-Serre spectral sequence. Thus we have $M^g = M^z$ and $g_M = \sharp(H/(\sigma-1)H) = \sharp H^G$ for H is assumed to be finite. Note that $I_GH = [\Gamma, H] \supset [\Gamma,[\Gamma,\Gamma]] =: \Gamma^{(3)}$ and consider the Milnor presentation concerning the nilpotent quotient of the link group G_L([Mi]):

$$(2.1)\quad G_L/G_L^{(3)} = \left\langle x_1,\dots,x_\mu \;\middle|\; \prod_{j\ne i}[x_i,x_j]^{\mathrm{lk}(K_i,K_j)} = 1\ (1\le i\le\mu),\ F^{(3)} = 1 \right\rangle$$

where x_i stands for a meridian around K_i and F is the free group generated by x_i's. We apply the Reidemeister-Schreier method ([MKS], 2.3) to the presentation (2.1) to obtain a presentation of the subgroup $\pi_1(Y)/G_L^{(3)}$ and that of $\pi_1(Y)/[G_L,\pi_1(Y)]$. To do so, we adjoin a generator u and a relation $u = x_1^{\lambda_1}\cdots x_\mu^{\lambda_\mu}$ to the presentation (2.1), where λ_i's are integers satisfying $a_1\lambda_1 + \cdots + a_\mu\lambda_\mu \equiv 1 \bmod n$, and take $1, u, \dots, u^{n-1}$ as a Schreier system of representatives for $G_L/\pi_1(Y)$. Then, taking the branch relation $x_i^{e_i} = 1$ $(1\le i\le\mu)$ into account and abelianizing, we derive the following presentation for H/I_GH:

$$(2.2)\ H/I_GH \simeq \left(\prod_{i=1}^{\mu}\mathbf{Z}/e_i\mathbf{Z}\right) \Big/ \langle(\lambda_1,\dots,\lambda_\mu)\rangle.$$

On the other hand, by the definition of Φ, the map Φ gives a surjective homomorphism

$$\begin{aligned} (2.3)\quad &H_1(M,\mathbf{Z}) \longrightarrow \\ &\mathrm{Ker}(H_1(X,\mathbf{Z})/\langle e_it_i(1\le i\le\mu)\rangle \longrightarrow G) \\ &\simeq \left\{ (\xi_i \bmod e_i) \in \prod_{i=1}^{\mu}\mathbf{Z}/e_i\mathbf{Z} \;\middle|\; \sum_{i=1}^{\mu} a_i\xi_i \equiv 0 \bmod n \right\} \end{aligned}$$

and we easily see $I_GH = (\sigma-1)H_1(M,\mathbf{Z}) \subset \mathrm{Ker}(\Phi)$. Since the right hand sides of (2.2) and (2.3) are isomorphic, $\mathrm{Ker}(\Phi)$ coincides with $(\sigma-1)H_1(M,\mathbf{Z})$.

Remarks. 1) Sakuma ([S2], Lemma 1) gave another description of $\mathrm{Ker}(\Phi)$ for any abelian M/S^3 using homology sequences. We can actually show that his description of $\mathrm{Ker}(\Phi)$ coincides with $[\Gamma,\Gamma]$, which yields another proof of our theorem. The idea to use the Reidemeister-Schreier method applied to the Milnor presentation for $G_L/G_L^{(3)}$ to study the structure of H/I_GH is due to K. Murasugi. In the case of strongly cyclic coverings, i.e., the case that $\psi(t_i)$ are the same generator of G, Murasugi also showed the author another geometric proof of our theorem by the computation using cycles on a Seifert surface of a link. We also refer to [S1], Theorem 7, [S3], Sublemma 15.4 and [MM] for related results.
2) Reznikov ([R2], 15) established, among other things, the genus theory for cyclic coverings with prime degree of any rational homology 3-spheres. His method is entirely different from ours.

3. We say that the *principal genus theorem* holds for a finite abelian covering M over S^3 if $M^g = M^z$. Our theorem in Section 1 says that the principal genus theorem holds for a cyclic M/S^3. For a non-cyclic covering $M \to S^3$, the principal genus theorem does not hold in general. We give here a concrete example. Let $L = K_1 \cup K_2$ be a 2-component link in S^3. Notations being as above, let $\psi : H_1(X, \mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ be a surjective homomorphism and consider the corresponding $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$-covering M of S^3 ramified over L. Assume that M is a rational homology 3-sphere. This is the case where L is a 2-bridge link $B(2\alpha, \beta)$ ($(2\alpha, \beta) = 1$, $-2\alpha < \beta < 2\alpha$, β odd) and M is the lens space $L(\alpha, \beta)$ ([BZ], Ch. 12). We first see that M^g must coincide with M. Then, by the same method as in our proof of the theorem, we can show $H/I_G H = \{0\}$ or $\mathbf{Z}/2\mathbf{Z}$ according to $\mathrm{lk}(K_1, K_2) \equiv 1$ or $0 \bmod 2$ respectively. Hence, we have $M^z = M^g$ if and only if $\mathrm{lk}(K_1, K_2) \equiv 1 \bmod 2$. For example, the principal genus theorem does not hold for the $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$-cover $L(\alpha, \beta)$ over S^1 ramified along $B(2\alpha, \beta)$ with even α, since $\mathrm{lk}(K_1, K_2) \equiv \alpha \bmod 2$ for $B(2\alpha, \beta) = K_1 \cup K_2$. We note that this is seen as an analog of Tate's criterion

$$\left(\frac{p_1}{p_2}\right) = -1 \quad \text{or} \quad \left(\frac{p_2}{p_1}\right) = -1,$$

i.e., $\mathrm{lk}_2(p_1, p_2) \equiv 1$ or $\mathrm{lk}_2(p_1, p_2) \equiv 1 \bmod 2$ in terms of [Mo1, 2], for a biquadratic field $\mathbf{Q}(\sqrt{p_1}, \sqrt{p_2})$ (p_1, p_2 being odd prime numbers) to enjoy the principal geneus theorem, equivalently, the Hasse norm principle over $\mathbf{Q}$ ([CF], p. 199 or p. 360; [Ra], §5, Ex. 1).

We refer to [R2], [T] for another approach to study the homology groups of $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$-coverings of 2-component links. We hope to discuss the material in detail together with more general cases in the future.

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