Boundedness of canonical Q-Fano 3-folds

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Abstract: We give an effective bound of the Gorenstein index of weak **Q**-Fano 3-folds, and prove the boundedness of the terminal **Q**-Fano 3-folds. Combined with [Bor99], this result implies furthermore the boundedness of the canonical **Q**-Fano 3-folds.

Key words: **Q**-Fano variety; terminal singularity; canonical singularity.

1. Introduction. In this paper, we will work over an algebraically closed field k of characteristic 0.

Definition 1.1. Let X be a normal projective variety and ε a positive number. We recall that X is said to have only *terminal* (resp. *canonical, klt, ε-lt*) singularities if all the discrepancies a of X satisfy $a > 0$ (resp. ≥ 0, > -1, $> -1 + \varepsilon$). X is called a *terminal* (resp. *canonical, klt, ε-lt*, etc.) ***Q**-Fano variety* if X has only terminal (resp. canonical, klt, ε-lt, etc.) sigularities and $-K_X$ is ample. By replacing 'ample' with 'nef and big', *terminal* (resp. *canonical, klt, ε-lt*, etc.) *weak **Q**-Fano varieties* are similarly defined.

Let $I(X)$ be the smallest positive integer I such that IK_X is Cartier; $I(X)$ is called the *Gorenstein index* of X.

We note that if X is a klt **Q**-Fano variety then $|-mK_X|$ is free for some $m > 0$. The induced birational morphism $X \to \overline{X}$ (resp. the target $\overline{X}$) is said to be the *anti-canonical morphism* (resp. *anti-canonical model*) of X.

Our main result is the following, which was announced in [KMM92c]:

Theorem 1.2. *Let X be a terminal weak **Q**-Fano 3-fold. Then the following hold.*

(1) $-K_X \cdot c_2(X) \geq 0$, *and hence* $I(X) | 24!$.
(2) *Assume further that the anti-canonical morphism $g : X \to \overline{X}$ does not contract any divisors. Then* $(-K_X)^3 \leq 6^3 \cdot (24!)^2$.
(3) *The terminal **Q**-Fano 3-folds are bounded, i.e. there is a morphism of schemes of finite type $f : \mathcal{X} \to S$ such that every geometric fiber of f is a terminal **Q**-Fano 3-fold and every terminal **Q**-Fano 3-fold appears as a geometric fiber of f.*

The bounds above are far from being sharp and the only meaning of such bounds is the effectiveness. Theorem 1.2 is a generalization of [Kaw92, Thm. 2], where 1.2 (1) and (2) were proved for **Q**-factorial X with $\rho(X) = 1$.

The proof of 1.2 (2) is similar to the ones of [KMM92a, Thm.] and [KMM92c, Thm. 0.2], where the technique of gluing rational curves plays a crucial role. For the proof of 1.2 (1), we need the theorem of [Bat92, 3.2 Thm.] which is a structure theorem of the cone of nef curves (see 2.2 below).

As for klt **Q**-Fano varieties, they do not form a bounded family. However A. Borisov [Bor96, Bor99] proved that the klt **Q**-Fano 3-folds with a fixed index form a bounded family. Combining 1.2 (1) with this result, we obtain the following.

Corollary 1.3. *The canonical **Q**-Fano 3-folds are bounded.*

There is a more general conjecture by V. Alexeev and A. Borisov:

Conjecture 1.4. *For arbitrary positive ε, ε-lt **Q**-Fano varieties are bounded.*

The conjecture in dimension 2 was proved by V. Alexeev [Ale94], and a simpler proof was given by V. Alexeev and S. Mori [AM95].

2. Preliminaries for 1.2 (1). We will recall definitions and results without proofs to prove (1).

Definition 2.1. Let X be a normal projective variety. Let $\mathrm{NE}^1(X)$ be the closed convex cone generated by effective **Q**-Cartier divisors $\subset \mathrm{N}^1(X) \otimes \mathbf{R}$, where $\mathrm{N}^1(X)$ is the group of numerical equivalence classes of **Q**-Cartier divisors. The dual cone $\mathrm{NM}_1(X) \subset \mathrm{N}_1(X) \otimes \mathbf{R}$ of $\mathrm{NE}^1(X)$ is called the *cone*

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of nef curves, where $\mathrm{N}_1(X)$ is the group of numerical equivalence classes of 1-cycles.

Theorem-Definition 2.2. *Let (X, Δ) be a projective $\mathbf{Q}$-factorial dlt pair of dimension at most 3 and H an ample Cartier divisor on X. Let ε be an arbitrary positive number, and let*

$$\mathrm{NE}_1^{\varepsilon}(X) := \{z \in \mathrm{NE}_1(X) | -(K_X + \Delta) \cdot z \leq \varepsilon H \cdot z\}.$$

Then there are a certain number, say $r(\varepsilon)$, of elements

$$[l_i] \in \mathrm{NM}_1(X) \cap \mathrm{N}_1(X)_{\mathbf{Z}} \setminus \mathrm{NE}_1^{\varepsilon}(X)$$

such that

$$\mathrm{NM}_1(X) + \mathrm{NE}_1^{\varepsilon}(X) = \sum_{i=1}^{r(\varepsilon)} \mathbf{R}_+[l_i] + \mathrm{NE}_1^{\varepsilon}(X).$$

When no l_i can be omitted in the equality, such a $\mathbf{R}_+[l_i]$ is called a coextremal ray. The following is the recipe for the coextremal rays.

Assume that we obtain a log Mori fiber space

$$g' : X' \to Y',$$

by running $(K_X + \Delta)$-MMP. Let $\phi : X \dashrightarrow X'$ be the natural birational map and

$$U' := \{x' \in X' | \phi^{-1} \text{ is an isomorphism at } x'\}.$$

Since $\mathrm{codim}(X' \setminus U') \geq 2$, *we can take a projective curve ${l_i}' \subset U'$ contained in a fiber of g. Let l_i be the strict transform of ${l_i}'$. Then $\mathbf{R}_+[l_i]$ is a coextremal ray for some ε. Conversely every coextremal ray is obtained by this procedure.*

We say that $g' : X' \to Y'$ is a log Mori fiber space associated to $\mathbf{R}[l_i]$.

Proof. In [Bat92, 3.2 Thm.], a proof is given for the assertion in case $(X, 0)$ is terminal. The proof works in our case. □

Corollary 2.3. *Under the notation and assumptions of* 2.2, *assume further that $-(K_X + \Delta)$ is ample. Then $\mathrm{NM}_1(X) = \sum_{i=1}^{r} \mathbf{R}_+[l_i]$.*

Theorem 2.4 (Thm. 6.1 of [Miy87])**.** *Let X be a normal projective variety which is smooth in codimension 2. Let $\mathcal{E}$ be a torsion free sheaf on X such that*

1. *$c_1(X)$ is a nef $\mathbf{Q}$-Cartier divisor, and*
2. *$\mathcal{E}$ is generically $(H_1, H_2, \ldots, H_{n-1})$-semi-positive for ample divisors H_i, i.e.*

$$c_1(\mathcal{L}) \cdot H_1 \cdot H_2 \cdot \cdots \cdot H_{n-1} \geq 0$$

for every quotient torsion-free sheaf $\mathcal{E} \to \mathcal{L}$. Then $c_2(\mathcal{E}) \cdot H_1 \cdot H_2 \cdot \cdots \cdot H_{n-2} \geq 0$.

Definition 2.5. Let X be a variety and $\pi_2 : \mathcal{C} \to S$ a flat family of irreducible projective curves in X over an irreducible base S. $\mathcal{C}$ is naturally contained in $X \times S$ so that π_2 is the restriction of the second projection $p_2 : X \times S \to S$.

$\mathcal{C}$ is said to be a *covering family* if $p_1(\mathcal{C})$ contains an open dense subset of X, where p_1 is the first projection $X \times S \to X$.

We denote by $\{C\}$ a covering family with a general element C. A point $x \in X$ is said to be a *fixed point* of $\{C\}$ if x belongs to all the members of $\{C\}$.

Definition-Proposition 2.6. *Let X be a variety and C a rational curve contained in* $\mathrm{Reg}\, X$. *We say that C is a free rational curve if $T_X^1|_C$ is semi-positive.*

Let $\{C\}$ be a covering family of rational curves on X such that $C \subset \mathrm{Reg}\, X$. Then a general C is a free rational curve.

We refer to [KMM92b, Cor.(1.3)] for the proof. The following formula is from [Kaw86, Lem. 2.2, 2.3] or [Rei87, (10.3)].

Theorem 2.7. *Let X be a projective terminal 3-fold. Then*

$$\chi(\mathcal{O}_X) = \frac{1}{24}(-K_X) \cdot c_2(X) + \frac{1}{24} \sum \left(r_i - \frac{1}{r_i}\right).$$

where r_i are indices of cyclic quotient terminal singularities obtained by deforming singularities of X locally.

3. Proofs of 1.2 (1) and (3).

Proof of "(1), (2)⇒ (3)". Let X be a terminal $\mathbf{Q}$-Fano 3-fold. By (1), $L := -(24!)K_X$ is an ample Cartier divisor which satisfies $(L^3) \leq 6^3 \cdot (24!)^2$ by (2). So there are a finite number of the possibilities of the two highest coefficients of the polynomial $P(t) := \chi(\mathcal{O}_X(tL))$. Hence by [KM83], there are also a finite number of the possibilities of $P(t)$ and by [Kol85, Thm. 2.1.3], the boundedness follows. □

We will prove (1) in the rest of this section.

Proof of 1.2 (1). Let X be a terminal weak $\mathbf{Q}$-Fano 3-fold. Then the $\mathbf{Q}$-factorialization $\pi : \tilde{X} \to X$ of X is a $\mathbf{Q}$-factorial terminal weak $\mathbf{Q}$-Fano 3-fold such that $-K_{\tilde{X}} = \pi^*(-K_X)$, $I(\tilde{X}) = I(X)$, the exceptional set of π is at most 1-dimensional. Thus we may assume that X is $\mathbf{Q}$-factorial by working on $\tilde{X}$ instead of X.

If $-K_X \cdot c_2(X) \geq 0$ holds, then $\sum(r_i - 1/r_i) \leq 24$ by 2.7 and $\chi(\mathcal{O}_X) = 1$ and hence 24! is divisible by the $I(X) = \mathrm{l.c.m.}\{r_i\}$. So it suffices to prove that

$-K_X \cdot c_2(X) \geq 0$.

Claim 3.1. *There is a* **Q**-*boundary* Δ *such that*

1. $-(K_X + \Delta)$ *is ample,*
2. (X, Δ) *is terminal, and*
3. *if the anti-canonical morphism* $g : X \to \overline{X}$ *of* X *contracts no divisors, then every irreducible component of* Δ *is movable.*

Proof. Let H (resp. $\overline{H}$) be a very ample divisor on X (resp. $\overline{X}$) such that $f^*\overline{H} - H$ is linearly equivalent to an effective divisor D which is very ample outside the exceptional set of g. We have only to put $\Delta := D/m$ for a sufficiently large natural number m. □

Thus by 2.3, we have $\mathrm{NM}_1(X) = \sum_{i=1}^{r} \mathbf{R}_+[l_i]$.

Let $\mathcal{E} := T^1_X$. Then $c_1(\mathcal{E})$ is nef. Hence by 2.4, it suffices to prove that $\mathcal{E}$ is generically (H_1, H_2)-semi-positive for ample divisors $H_i (i = 1, 2)$. Since $H_1 \cdot H_2 \in \mathrm{NM}_1(X)$, we have only to prove $c_1(\mathcal{L}) \cdot l_i \geq 0$ for each i and every surjection $\mathcal{E} \to \mathcal{L}$ to an arbitrary torsion free sheaf $\mathcal{L}$.

We can assume that l_i is obtained as stated in 2.2, with the same notation and assumptions and we fix i till the end of this section. Furthermore let U be the open set of X corresponding to U' and Δ' the strict transform of Δ on X'.

We extend $T^1_U \to \mathcal{L}|_U/(\mathrm{tor})$ to $T^1_{X'} \to \mathcal{L}'$ via $U \simeq U' \to X'$. Since $X \to X'$ is an isomorphism on $U \supset l_i$, we have $c_1(\mathcal{L}) \cdot l_i = c_i(\mathcal{L}') \cdot l'_i$. Thus our assertion is equivalent to $c_1(\mathcal{L}') \cdot {l_i}' \geq 0$. Since $\rho(X'/Y') = 1$, we may replace $\{{l_i}'\}$ by a covering family of rational curves $\{l\}$ of X' such that l is contained in a fiber of $X' \to Y'$.

Let $\pi : \tilde{X}' \to X'$ be the resolution of X' and $\tilde{l}$ the strict transform of l. Since X' is **Q**-factorial, $((\wedge^r \mathcal{L})^{\otimes n})^{**}$ is invertible for some n, where r is the rank of $\mathcal{L}$. Note that we have the natural map

$$\mathcal{S}^n(\wedge^r T^1_{\tilde{X}'}) \to \pi^*(((\wedge^r \mathcal{L})^{\otimes n})^{**})$$

and its restriction to $\tilde{l}$ has a finite cokernel. Hence by 2.6, $c_1(\mathcal{L}) \cdot l = \pi^* c_1(\mathcal{L}) \cdot l' \geq 0$. This completes the proof of 1.2 (1). □

4. Preliminaries for 1.2 (2).

Lemma 4.1. *Let* X *be an* n*-dimensional projective variety and* x *a closed point with multiplicity* r*. Let* D *be a nef and big* **Q**-*Cartier divisor on* X *and* $\{l\}$ *a covering family of curves containing* x *such that* $D \cdot l \leq d$*. Then* $D^n \leq rd^n$*.*

Proof. Though the proof of [KMM92a, Cor. 1] was for smooth X, it works in our case with obvious changes. □

Theorem 4.2 (Gluing lemma)**.** *Let* $C_1 \cup C_2$ *be the union of* $C_i \simeq \mathbf{P}^1$ *intersecting at one point, which is an ordinary double point of* $C_1 \cup C_2$*. Let* P *be a point on* $C_2 \setminus C_1$*. Let* X *be a variety and* $\nu : C_1 \cup C_2 \to X$ *be a morphism such that* $l_i := \nu(C_i)$ *are free rational curves contained in* $\mathrm{Reg}\, X$ *and* $x := \nu(P) \notin l_1$*. Then* ν *deforms to a morphism* $\nu' : C \to X$ *from* $C \simeq \mathbf{P}^1$ *such that* $x \in \nu'(C)$*. Furthermore we can choose* ν' *so that* $\nu'(C)$ *is a free rational curve.*

Proof. This is a special case of [KMM92b, (1.8) Cor.]. The last assertion is easy since $C \simeq \mathbf{P}^1$. □

Remark 4.3. For a projective variety X, we use 4.2 as follows.

(1) Let $\{l_i\} (i = 1, 2)$ be covering families of free rational curves. We fix a closed point x and choose l_1, l_2 so that $x \in l_1 \setminus l_2$ and $l_1 \cap l_2$ contains a point, say y. Let $C_1 \cup C_2$ be as in 4.2 and $\mu : C_1 \cup C_2 \to l_1 \cup l_2$ the morphism such that $\mu|_{C_i}$ are the normalizations. Let ν be the composition of μ and the embedding $l_1 \cup l_2 \hookrightarrow X$. We can apply 4.2 for μ with a point $x \in l_2 \setminus l_1$ fixed. Then we obtain a new covering family of free rational curves $\{m\}$.

We say that $\{m\}$ is obtained by *gluing* $\{l_1\}$ and $\{l_2\}$ (with x fixed).

(2) Let $l_1, \ldots, l_r$ $(r \geq 3)$ be free rational curves such that $l_1, \ldots, l_r$ form a linear chain and $x \in l_1 \setminus (l_2 \cup \cdots \cup l_r)$. Let $z \in l_{r-2} \cap l_{r-1}$. We can glue l_{r-1} and l_r with z fixed into $m \ni z$ and get a linear chain $l_1, \ldots, l_{r-2}, m$ of $r - 1$ free rational curves.

Construction-Proposition 4.4. *Under the notation and assumptions of* 2.5*, let* $\mathcal{C} = \{l\}$ *be a covering family of rational curves of* X *such that* $l \subset \mathrm{Reg}\, X$*. Shrinking the parameter space* S*, we will assume all the members of* $\{l\}$ *are free rational curves.*

Let $\pi_1 = p_1|_{\mathcal{C}} : \mathcal{C} \to X$ *and* $s \in p_1(\mathcal{C})$ *a (smooth) closed point of* X*. Then we construct constructible subsets* $V^k_{\{l\}}(x) \subset X$ *inductively:*

$$V^0_{\{l\}}(x) := \{x\},$$
$$V^{k+1}_{\{l\}}(x) := \pi_1 \pi_2^{-1} \pi_2 \pi_1^{-1} V^k_{\{l\}}(x).$$

Let $V_{\{l\}}(x) := \cup_k V^k_{\{l\}}(x)$ *and, for a subset* W*, let* $\overline{W}$ *denote the closure. Then we have*

$$\overline{V^k_{\{l\}}}(x) = \overline{V^{k+1}_{\{l\}}}(x) \quad (\forall k \geq \max_{n \geq 0} \dim V^n_{\{l\}}(x)).$$

The proof of [KMM92c, Lem. 1.3] works with no changes.

5. Proof of 1.2 (2)-the case $\rho(X) = 1$. In this section, we prove the special case of 1.2 (2):

Theorem 5.1. *Let X be a $\mathbf{Q}$-factorial terminal $\mathbf{Q}$-Fano 3-fold with $\rho(X) = 1$. Then*

$$(-K_X)^3 \leq 6^3 \cdot (24!)^2.$$

This was proved in [Kaw92, Thm. 2] with a possibly different bound. Here we give an alternate proof by the method of [KMM92a].

Proof. By [MM86, Thm. 5], there is a covering family of rational curves $\{l\}$ such that $-K_X \cdot l \leq 6$.

If $\{l\}$ has a fixed point x, then by 4.1, we have $(-K_X)^3 \leq 6^3 \mathrm{mult}_x X$. Since the canonical cover of (X, x) is at worst a cDV singularity, we have

$$\mathrm{mult}_x X \leq 2 \cdot (\mathrm{index}_x X)^2.$$

By virtue of 1.2 (1), we have

$$\mathrm{index}_x X \leq 24!.$$

Hence $(-K_X)^3 \leq 6^3 \cdot (24!)^2$ in this case.

If $\{l\}$ has no fixed points, the proof is the same as the one of [KMM92a, Thm.].

Claim 5.2. $\overline{V_{\{l\}}(x)} = X$ *for a general closed point $x \in X$.*

Proof of 5.2. Let U be an open set containing $p_1(\mathcal{C})$. Similarly to $V^k_{\{l\}}(x)$, we define constructible subsets $V^k_{\{\mathcal{C}\}}(U) \subset X \times U$ as follows:

$$V^0_{\{\mathcal{C}\}}(U) := \{(x, x) \in X \times U\},$$
$$V^{k+1}_{\{\mathcal{C}\}}(U) := \Pi_1 \Pi_2^{-1} \Pi_2 \Pi_1 V^k_{\{\mathcal{C}\}}(U),$$

where $\Pi_i = \pi_i \times \mathrm{id}$. Let $V_{\{\mathcal{C}\}}(U) := \cup_k V^k_{\{\mathcal{C}\}}(U)$ and q_i the i-th projection of $X \times U$ to its i-th factor $(i = 1, 2)$. Then since $V^k_{\{\mathcal{C}\}}(U)$ is constructible, we can choose an open dense subset $U' \subset U$ so that $\overline{V}^k_{\{\mathcal{C}\}}(U)|_{q_2^{-1}(x)} = \overline{V}^k_{\{l\}}(x)$ and hence

$$\overline{V}_{\{\mathcal{C}\}}(U)|_{q_2^{-1}(x)} = \overline{V}_{\{l\}}(x)$$

for all $x \in U'$ by 4.4.

Assume that the claim fails. Then

$$d := \dim \overline{V}_{\{\mathcal{C}\}}(U) - \dim X < 3.$$

Let $W \subset U'$ be a general complete intersection of codimension $d + 1$. Then $D := \overline{q_1 \circ q_2^{-1}(W)}$ is of codimension 1 in X. Note that $\overline{D \setminus \cup_{x \in W} V_{\{l\}}(x)}$ is of codimension ≥ 2 in X. So by [Kol96, p.115, 3.7 Proposition], we may assume that l is disjoint from $\overline{D \setminus \cup_{x \in W} V_{\{l\}}(x)}$. If l intersects $V_{\{l\}}(x)$ for some x, then $l \subset V_{\{l\}}(x)$ by the definition of $V_{\{l\}}(x)$. Hence we may assume that $D \cap l = \phi$. But this is a contradiction since the divisorial part of D is ample. □

By 5.2 and 4.4, there is a covering family of rational curves $\{l'\}$ with a fixed point x such that $-K_X \cdot l' \leq 3 \times 6$. Hence by 4.1, $(-K_X)^3 \leq 18^3$ in this case. □

6. Proof of 1.2 (2). As we did at the beginning of Section 3, we may assume that X is a $\mathbf{Q}$-factorial terminal weak $\mathbf{Q}$-Fano threefold to pvove (2).

Let Δ be as in 3.1 and $g : X' \to Y'$ an arbitrary end result of the $(K_X + \Delta)$-MMP. By the assumption of (2), every component of Δ is movable by 3.1.3. Hence (X', Δ') is also terminal, where Δ' is the strict transform of Δ.

First we treat the case where $\rho(X') = 1$ for some X'. Then X' is a $\mathbf{Q}$-factorial terminal $\mathbf{Q}$-Fano 3-fold with $\rho(X') = 1$. Hence $(-K_{X'})^3 \leq 6^3 \cdot (24!)^2$ by 5.1. Since $h^0(-mK)$ does not decrease under $(K_X + \Delta)$-MMP for any m, we have $h^0(-mK_X) \leq h^0(-mK_{X'})$. So by the Riemann-Roch theorem and the Kodaira-Kawamata-Viehweg vanishing theorem, we have $(-K_X)^3 \leq (-K_{X'})^3$. Hence $(-K_X)^3 \leq 6^3 \cdot (24!)^2$ in this case.

We are left with the case $\rho(X') \geq 2$ for all X'. Thus, for any coextremal ray $\mathbf{R}_+[l]$, the target Y' of the associated log Mori fiber space $g : X' \to Y'$ is not a point. Since X' is terminal, general fibers of g are smooth rational curves or smooth del Pezzo surfaces. Hence we can take as l smooth free rational curves such that $-K_X \cdot l \leq 3$.

Since $\dim \mathrm{NM}_1(X) \otimes \mathbf{R} = \rho(X) \geq 2$, we know that $\mathrm{NM}_1(X)$ has at least 2 coextremal rays. Let $\mathbf{R}_+[l_1]$ and $\mathbf{R}_+[l_2]$ be two of them. Let x be a general point of X and $\overline{V_i} := \overline{V_{\{l_i\}}}(x)$. It suffices to treat three cases:

(i) $\dim \overline{V_i} = 1$ for $i = 1, 2$,

(ii) $\dim \overline{V_1} = 2$ and $\dim \overline{V_2} \leq 2$, and

(iii) $\dim \overline{V_1} = 3$.

In case (i), the dimension of a fiber of g_i is 1 and hence $-K_X \cdot l_i \leq 2$. By gluing $\{l_1\}$ and $\{l_2\}$, we obtain a new covering family $\{l_3\}$ such that $-K_X \cdot l_3 \leq 4$ and $\dim \overline{V^1_{\{l_3\}}}(x) > \dim \overline{V^1_{\{l_1\}}}(x) = 1$. Let $\overline{V^1_3} := \overline{V^1_{\{l_3\}}}(x)$.

We first treat the case $\dim \overline{V^1_3} = 2$. Since $\overline{V^1_3} \in \mathrm{NE}^1(X)$, there is a coextremal ray $\mathbf{R}_+[m]$ such that $\overline{V^1_3} \cdot m > 0$. As seen above, we can take as m smooth free rational curves with $-K_X \cdot m \leq 3$. We can assume $l_3 \cap m \neq \emptyset$. By gluing $\{l_3\}$ and $\{m\}$ with x fixed, we obtain smooth free rational curves $l_4 \ni x$ such that $-K_X \cdot l_4 \leq 7$ and $\dim \overline{V^1_{\{l_4\}}} > \dim \overline{V^1_3} = 2$. Hence by 4.1, we have $(-K_X)^3 \leq 7^3$.

If $\dim \overline{V^1_3} = 3$, we have $(-K_X)^3 \leq 4^3$ by 4.1.

In case (ii), by gluing $\{l_1\}$ and $\{l_1\}$ and $\{l_2\}$, we obtain a new covering family $\{l_3\}$ such that $-K_X \cdot l_3 \leq 9$ and $\dim \overline{V^1_{\{l_3\}}}(x) = 3$. Hence by 4.1, we have $(-K_X)^3 \leq 9^3$.

In case (iii), we have $(-K_X)^3 \leq (3 \times 3)^3$ by 4.1.

Hence we have $(-K_X)^3 \leq 6^3 \cdot (24!)^2$, and the proof of (2) is finished.

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