Fan's inequalities for vector-valued multifunctions

By Pando Grigorov GEORGIEV*) and Tamaki TANAKA**)

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Abstract: We present four variants of Fan's type inequality for vector-valued multifunctions in topological vector spaces with respect to a cone preorder in the target space, when the functions and the cone possess various kinds of semicontinuity and convexity properties. Using the classical scalar Fan inequality, we prove directly a two-function result of Simons, which is used to establish our main tool for proving the presented results.

Key words: Fan's inequality; vector-valued multifunctions; semicontinuous mappings; quasiconvex functions.

1. Introduction. Fan's inequality is one of the main tools in the nonlinear analysis. It is equivalent to other main theorems in nonlinear analysis, like Brouwer's fixed point theorem, Knaster-Kuratowski-Mazurkiewicz theorem, etc. (see for instance [2]). As an analytical instrument, in many situations it is more appropriate and applicable than the other main theorems.

In this paper we show four kinds of vector-valued Fan's type inequality for multifunctions. One of them (Theorem 3.1) generalizes the main result of Ansari-Yao in [1], namely, the existence result in the so-called Generalized Vector Equilibrium Problem. Any of our Theorems 3.1–3.4 implies the classical Fan inequality, while the main result in [1] does not imply it in its full generality, but only for continuous functions. Our proofs are quite different from that one in [1] and are based on the classical scalar Fan inequality. More precisely, in the proofs we use a new result (see Theorem 2.3) which follows from a two-function result of Simons [7, Theorem 1.2] (used in [7] to derive Fan's inequality), which we prove directly by Fan's inequality. For a simple proof of the classical Fan inequality, based on Brouwer's fixed point theorem, we refer to [3] and [8].

Our main tool in this paper (Theorem 2.3) is a slightly more general form of a two-function result of Simons [7, Corollary 1.6] and as a consequence of our results, it implies the classical Fan inequality.

The proofs of the main results (Theorems 3.1–3.4) use Theorem 2.3 for special scalar functions possessing semicontinuity and convexity properties, inherited by the semicontinuity and the convexity properties of the multifunctions. These proofs will be published in [4] and elsewhere (Nonlinear Analysis, TMA).

2. Fan's inequality and a new two-function result. Firstly we recall the classical scalar Fan inequality and prove that it implies a two-function result of Simons (namely [7, Theorem 1.2]), which is used in the sequel to prove the main tool for proving the multivalued versions of Fan's inequality (Theorems 3.1–3.4).

Theorem 2.1 (Fan). *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \mathbf{R}$ be quasiconcave in its first variable and lower semicontinuous in its second variable. Then*

$$\min_{y\in X} \sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

Theorem 2.2 (Simons [7, Theorem 1.2]). *Let Z be a nonempty compact convex subset of a topological vector space, $f : Z \times Z \to \mathbf{R}$ lower semicon-*

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*) Department of Mathematics and Informatics, University of Sofia, 5 James Bourchier Blvd., 1126 Sofia, Bulgaria. Current address: Laboratory for Advanced Brain Signal Processing, Brain Science Institute, The Institute of Physical and Chemical Research (RIKEN), 2-1, Hirosawa, Wako, Saitama 351-0198.

**) Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki, Aomori 036–8561.

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tinuous in its second variable, $g : Z \times Z \to \mathbf{R}$ *quasiconcave in its first variable, and* $f \le g$ *on* $Z \times Z$. *Then*

$$\min_{y\in Z}\sup_{x\in Z} f(x,y) \le \sup_{z\in Z} g(z,z).$$

Proof. Define the function $\operatorname{co} f$ as a quasiconcave envelope of f with respect to the first variable:

$$\operatorname{co} f(x,y) := \sup\Big\{\min_{i\in\{1,\dots,n\}} f(x_i,y) : x = \sum_{i=1}^{n}\lambda_i x_i, x_i \in Z, \lambda_i \ge 0, \sum_{i=1}^{n}\lambda_i = 1, n \in \mathbf{N}\Big\},$$

where $\mathbf{N}$ is the set of the natural numbers. This function satisfies the conditions of Fan's inequality and applying the latter, we obtain the result. □

Now we prove our main tool in this paper. Its proof is similar to that one of [7, Corollary 1.6].

Theorem 2.3. *Let* X *be a nonempty compact convex subset of a topological vector space,* $a : X \times X \to \mathbf{R}$ *lower semicontinuous in its second variable,* $b : X \times X \to \mathbf{R}$ *quasiconvex in its second variable, and*

$$x, y \in X \text{ and } a(x,y) > 0 \Rightarrow b(y,x) < 0.$$

Suppose that $\inf_{x\in X} b(x,x) \ge 0$. *Then there exists* $z \in X$ *such that* $a(x,z) \le 0$ *for all* $x \in X$.

Proof. Define

$$f(x,y) = 1 \quad \text{if } a(x,y) > 0 \text{ and}$$
$$f(x,y) = 0 \quad \text{otherwise.}$$

Analogically define

$$g(x,y) = 1 \quad \text{if } b(y,x) < 0 \text{ and}$$
$$g(x,y) = 0 \quad \text{otherwise.}$$

These functions satisfy the conditions of Theorem 2.2, and applying it, we obtain the result. □

3. Set-valued Fan's inequalities. Further let E and Y be topological vector spaces and F, $C : E \to 2^Y$ two multivalued mappings and let for every $x \in E$, $C(x)$ be a closed convex cone with nonempty interior. We introduce two types of cone-semicontinuity for set-valued mappings, which are regarded as extensions of the ordinary lower semicontinuity for real-valued functions; see [5].

Denote $B(x) = (\operatorname{int} C(x)) \cap (2S \setminus \overline{S})$ (an open base of $\operatorname{int} C(x)$), where S is a neighborhood of 0 in Y, and define the function

$$h(k,x,y) = \inf\{t : y \in tk - C(x)\},$$

Note that $h(k,x,\cdot)$ is positively homogeneous and subadditive for every fixed $x \in E$ and $k \in \operatorname{int} C(x)$. Moreover, we use the following notations

$$h(k,y) = \inf\{t : y \in tk - C\},$$

and $B = C \cap (2S \setminus \overline{S})$, where C is a convex closed cone and S is a neighborhood of 0 in Y. Note again that $h(k,\cdot)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$.

Firstly, we prove some inherited properties from cone-semicontinuity.

Definition 3.1. Let $\hat{x} \in E$. The multifunction F is $C(\hat{x})$*-upper semicontinuous* at x_0, if for every $y \in C(\hat{x}) \cup (-C(\hat{x}))$ such that $F(x_0) \subset y + \operatorname{int} C(\hat{x})$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \operatorname{int} C(\hat{x})$ for every $x \in U$. If Y is a Banach space, we shall say that F is $(-C)^c$*-upper semicontinuous* at x_0, if for any $\varepsilon > 0$ and $k \in C$ such that $(k + \varepsilon B_Y - C) \cap F(x_0) = \emptyset$, there exists $\delta > 0$ such that $(k + \varepsilon B_Y - C) \cap F(x) = \emptyset$ for every $x \in B(x_0;\delta)$.

Definition 3.2. Let $\hat{x} \in E$. The multifunction F is $C(\hat{x})$*-lower semicontinuous* at x_0, if for every open V such that $F(x_0) \cap V \ne \emptyset$, there exists an open $U \ni x_0$ such that $F(x) \cap (V + \operatorname{int} C(\hat{x})) \ne \emptyset$ for every $x \in U$. If Y is a Banach space, we shall say that F is $C(\hat{x})$*-lower semicontinuous* at x_0, if for any $\varepsilon > 0$ and $y_0 \in F(x_0)$ there exists an open $U \ni x_0$ such that $F(x) \cap (y_0 + \varepsilon B_Y + C(\hat{x})) \ne \emptyset$ for every $x \in U$, where B_Y denotes the open unit ball in Y.

Remark 3.1. In the two definitions above, the corresponding notions for single-valued function are equivalent to the ordinary one of lower semicontinuity for real-valued function whenever $Y = \mathbf{R}$ and $C = [0,\infty)$. When the cone $C(\hat{x})$ consists only of the zero of the space, the notion in Definition 3.2 coincides with that of lower semicontinuous set-valued mapping. Moreover, it is equivalent to the cone-lower semicontinuity defined in [5], based on the fact of $V + \operatorname{int} C(\hat{x}) = V + C(\hat{x})$; see [9, Theorem 2.2].

Lemma 3.1. *Suppose that multifunction* $W : E \to 2^Y$ *defined as* $W(x) = Y \setminus \operatorname{int} C(x)$ *has a closed graph. If the multifunction* F *is* $(-C(x))$*-upper semicontinuous at* x *for each* $x \in E$, *then the function* $\varphi_1|_X$ *(the restriction of*

$$\varphi_1(x) := \inf_{k\in B(x)}\sup_{y\in F(x)} h(k,x,y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P);

(P) *for every $x \in X$ there exists an open $U \ni x$ such that the set $F(U \cap X)$ is precompact in Y, that is, $\overline{F(U \cap X)}$ is compact.*

If the mapping C is constant-valued, then φ_1 is upper semicontinuous.

Lemma 3.2. *Suppose that the multifunction F is $-C(x)$-lower semicontinuous for each $x \in E$ and the multifunction $W : E \to 2^Y$ defined by $W(x) = Y \setminus \operatorname{int} C(x)$ has a closed graph. Then the function $\varphi_2|_X$ (the restriction of*

$$\varphi_2(x) := \inf_{k \in B(x)} \inf_{y \in F(x)} h(k, x, y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P). *If the mapping C is constant-valued, then φ_2 is upper semicontinuous.*

Lemma 3.3. *Suppose that Y is a Banach space and the multifunction $F : E \to 2^Y$ is $(-C)^c$-upper semicontinuous and locally bounded (it means that for every point $x_0 \in E$ there exists an open set $U \ni x_0$ and $p > 0$ such that $F(x) \subset pB_Y$ for every $x \in U$, where B_Y denotes the open unit ball in Y). Suppose that the multifunction C has a closed graph and the cone $C(x)$ has a compact base $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ for every x. Then the function φ_2 is lower semicontinuous.*

Lemma 3.4. *Suppose that Y is a Banach space and the multifunction $F : E \to 2^Y$ is $C(x)$-lower semicontinuous for each $x \in E$ and locally bounded. Suppose that the multifunction C has a closed graph and the cone $C(x)$ has a compact base $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ for every x. Then the function φ_1 is lower semicontinuous.*

Next, we show some inherited properties from cone-quasiconvexity.

Definition 3.3. A multifunction $F : E \to 2^Y$ is called *C-quasiconvex*, if the set $\{x \in E : F(x) \cap (a - C) \neq \emptyset\}$ is convex for every $a \in Y$. If $-F$ is C-quasiconvex, then F is said to be C-quasiconcave, which is equivalent to $(-C)$-quasiconvex mapping.

Remark 3.2. The above definition is exactly that of *Ferro type* (-1)-*quasiconvex* mapping in [6, Definition 3.5].

Definition 3.4. A multifunction $F : E \to 2^Y$ is called (in the sense of [6, Definition 3.6])

(a) *type*-(v) *C-properly quasiconvex* if for every two points x_1, $x_2 \in X$ and every $l \in [0, 1]$ we have either $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$;

(b) *type*-(iii) *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$.

If $-F$ is type-(v) [resp. type-(iii)] C-properly quasiconvex, then F is said be type-(v) [resp. type-(iii)] C-properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] $(-C)$-properly quasiconvex mapping.

Remark 3.3. The convexity of (a) above is exactly that of *C-quasiconvex-like* multifunction in [1].

Lemma 3.5. *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-properly quasiconvex, then the function*

$$\psi_1(x) := \inf_{k \in B} \sup_{y \in F(x)} h(k, y)$$

is quasiconvex.

Lemma 3.6. *If F is C-quasiconvex, then for every $k \in B$ the function*

$$\psi_2(x; k) := \inf\{h(k, y) : y \in F(x)\}$$

is quasiconvex.

Lemma 3.7. *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-properly quasiconcave, then the function $\psi_2(x; k)$ is quasiconcave, where $k \in \operatorname{int} C$.*

Lemma 3.8. *If the multifunction $F : E \to 2^Y$ is type-*(iii) *C-properly quasiconcave, then the function*

$$\psi_1(x; k) := \sup\{h(k, y) : y \in F(x)\}$$

is quasiconcave, where $k \in \operatorname{int} C$.

Now we state the main results in this paper. The following theorem is a generalization of that one in [1]. The main difference between our result and that one in [1] is the condition (iii), but it allows us to recover the classical Fan inequality, when Y is the real line. The result in [1] recovers it only for continuous functions.

Theorem 3.1. *Let K be a nonempty convex subset of a topological vector space E, Y be a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for every $x \in K$, $C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$;*

(ii) *$W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus \operatorname{int} C(x)$, and the graph of W is closed in $K \times Y$;*

(iii) *for every* $x,\ y \in K$, $F(\cdot, y)$ *is* $C(x)$*-upper semicontinuous at* x *with closed values on* K *and if the mapping* C *is not constant-valued, then the mapping* $F(\cdot, y)$ *maps the compact subsets of* K *into precompact subsets of* Y;

(iv) *there exists a multifunction* $G : K \times K \to 2^Y$ *such that*

(a) *for every* $x \in K$, $G(x, x) \not\subset \operatorname{int} C(x)$,

(b) *for every* $x,\ y \in K$, $F(x, y) \subset \operatorname{int} C(x)$ *implies* $G(x, y) \subset \operatorname{int} C(x)$,

(c) $G(x, \cdot)$ *is type-*(v) $C(x)$*-properly quasiconcave on* K *for every* $x \in X$,

(d) $G(x, y)$ *is compact, if* $G(x, y) \subset \operatorname{int} C(x)$;

(v) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x, y) \subset \operatorname{int} C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \not\subset \operatorname{int} C(x), \textit{for all } y \in K\}$$

is a nonempty and compact subset of D.

Theorem 3.2. *Let* K *be a nonempty convex subset of a topological vector space* E, Y *a topological vector space, and* $F : K \times K \to 2^Y$ *a multifunction. Assume that*

(i) $C : K \to 2^Y$ *is a multifunction such that for every* $x \in K$, $C(x)$ *is a closed convex cone in* Y *with* $\operatorname{int} C(x) \neq \emptyset$;

(ii) $W : K \to 2^Y$ *is a multifunction defined as* $W(x) = Y \setminus \operatorname{int} C(x)$, *for every* $x \in K$ *such that the graph of* W *is closed in* $K \times Y$;

(iii) *for every* $x,\ y \in K$, $F(\cdot, y)$ *is* $C(x)$*-lower semicontinuous with closed values on* K *and if the mapping* C *is not constant-valued, then the mapping* $F(\cdot, y)$, *for every* $y \in K$, *maps the compact subsets of* K *into precompact subsets of* Y;

(iv) *there exists a multifunction* $G : K \times K \to 2^Y$ *such that*

(a) *for every* $x \in K$, $G(x, x) \cap \operatorname{int} C(x) = \emptyset$,

(b) *for every* $x,\ y \in K$, $F(x, y) \cap \operatorname{int} C(x) \neq \emptyset$ *implies* $G(x, y) \cap \operatorname{int} C(x) \neq \emptyset$,

(c) $G(x, \cdot)$ *is* $C(x)$*-quasiconcave on* K *for every* $x \in K$;

(v) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x, y) \cap \operatorname{int} C(x) \neq \emptyset$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \cap \operatorname{int} C(x) = \emptyset, \textit{for all } y \in K\}$$

is a nonempty and compact subset of D.

Theorem 3.3. *Let* K *be a nonempty convex subset of a topological vector space* E, Y *a Banach space, and* $F : K \times K \to 2^Y$ *a multifunction. Assume that*

(i) $C : K \to 2^Y$ *is a multifunction with a closed graph and* $C(x)$ *is a closed convex cone with a compact base* $B(x) = (2\overline{B_Y} \setminus B_Y) \cap C(x)$ *for every* x;

(ii) *for every* $y \in K$, $F(\cdot, y)$ *is* $(-C)^c$*-upper semicontinuous and locally bounded*;

(iii) *there exists a multifunction* $G : K \times K \to 2^Y$ *such that*

(a) *for every* $x \in K$, $G(x, x) \cap (-C(x)) \neq \emptyset$,

(b) *for every* $x,\ y \in K$, $F(x, y) \cap (-C(x)) = \emptyset$ *implies* $G(x, y) \cap (-C(x)) = \emptyset$,

(c) $G(x, \cdot)$ *is type-*(v) $C(x)$*-properly quasiconcave on* K *for every* $x \in K$;

(v) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x, y) \cap (-C(x)) = \emptyset$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \cap (-C(x)) \neq \emptyset, \textit{for all } y \in K\}$$

is a nonempty and compact subset of D.

Theorem 3.4. *Let* K *be a nonempty convex subset of a topological vector space* E, Y *a Banach space, and* $F : K \times K \to 2^Y$ *a multifunction. Assume that*

(i) $C : K \to 2^Y$ *is a multifunction with a closed graph such that C(x) is a closed convex cone with a compact base* $B(x) = (2\overline{B}_Y \setminus B_Y) \cap C(x)$ *for every* x;

(ii) *for every* $x, y \in K$, $F(\cdot, y)$ *is* $C(x)$*-lower semicontinuous and locally bounded*;

(iii) *there exists a multifunction* $G : K \times K \to 2^Y$ *such that*

(a) *for every* $x \in K$, $G(x, x) \subset -C(x)$,

(b) *for every* $x,\ y \in K$, $F(x, y) \not\subset -C(x)$ *implies* $G(x, y) \not\subset -C(x)$,

(c) $G(x, \cdot)$ *is type-*(iii) $C(x)$*-properly quasiconcave on* K *for every* $x \in K$;

(iv) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x, y) \not\subset -C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \subset -C(x), \textit{for all } y \in K\}$$

is a nonempty and compact subset of D.

As a corollary from any of Theorems 3.1, 3.2, 3.3, and

3.4, we obtain that Theorem 2.3 implies the scalar Fan inequality.

Idea of the proofs (for example, of Theorem 3.1): put

$$a(x, y) := -\inf_{k \in B(y)} \sup_{z \in -F(y,x)} h(k, y, z)$$

$$b(x, y) := \inf_{k \in B(x)} \sup_{z \in -G(x,y)} h(k, x, z)$$

and apply Theorem 2.3 for the convex hull of D and finitely many points of K. In such a way we obtain a finite intersection property of certain family of sets and using compactness argument, we prove the full statement.

The other theorems can be proved by using the same idea, but for different functions a and b.

References

[1] Ansari, Q. H., and Yao, J.-C.: An existence result for the generalized vector equilibrium problem. Appl. Math. Lett., **12**, 53–56 (1999).

[2] Aubin, J.-P., and Ekeland, I.: Applied Nonlinear Analysis. Wiley Interscience, New York (1984).

[3] Aubin, J.-P., and Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990).

[4] Georgiev, P. G., and Tanaka, T.: Vector-valued set-valued variants of Ky Fan's inequality. J. Nonlinear and Convex Analysis, **1(3)**, 245–254 (2000).

[5] Kimura, Y., Tanaka, K., and Tanaka, T.: On semicontinuity of set-valued maps and marginal functions. Nonlinear Anlysis and Convex Analysis—Proceedings of the International Conference (eds. Takahashi, W., and Tanaka, T.). World Scientific, Singapore, pp. 181–188 (1999).

[6] Kuroiwa, D., Tanaka, T., and Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Analysis, TMA, **30(3)**, 1487–1496 (1997).

[7] Simons, S.: Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems. Proc. Symp. Pure Math., **45(2)**, 377–392 (1986).

[8] Takahashi, W.: Nonlinear variational inequalities and fixed point theorems. J. Math. Soc. Japan, **28(1)**, 168–181 (1976).

[9] Tanaka, T., and Kuroiwa, D.: Another observation on conditions assuring $\operatorname{int} A + B = \operatorname{int}(A + B)$. Appl. Math. Lett., **7(1)**, 19–22 (1994).