Intersection matrix of a generalized Airy function in terms of skew-Schur polynomials

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Abstract: A duality is introduced between a pair of polynomial twisted de Rham cohomology groups associated with a generalized Airy function in several variables. Natural bases of the twisted de Rham groups are constructed in terms of Schur polynomials. Then the intersection matrix relative to these bases is calculated explicitly in terms of skew-Schur polynomials.

Key words: Generalized Airy function; twisted de Rham cohomology group; duality; intersection matrix; Schur polynomial; skew-Schur polynomial.

1. Introduction. The classical Airy function in single-variable is defined by a one-dimensional complex integral:

$$\mathrm{Ai}(a) = \int_c e^{(1/3)t^3+at}\,dt \quad (a \in \mathbf{C}),$$

where c is a cycle chosen in such a way that the integrand is exponentially decreasing at infinity along c (see Fig. 1). The Airy function is an important special function arising in mathematical optics (see Airy [1]). A generalization of the Airy function into several variables was introduced by Gel'fand, Retakh and Serganova [4], and was studied in some depth by Kimura [7, 8].

In studying hypergeometric functions in their broadest sense, including generalized Airy functions, it is important to investigate the structure of rational twisted de Rham cohomology groups associated with them. This is because, from the viewpoint of Euler integral representations, a hypergeometric function is defined to be an integral of a closed twisted differential form along a twisted cycle (see e.g., Aomoto and Kita [2]).

Kimura [7] constructed a basis of the polynomial twisted de Rham cohomology group for a generalized Airy function in terms of Schur polynomials (without proof). A standard fact about the cohomology ring of a Grassmannian manifold tells us that his conjectural basis is actually a basis.

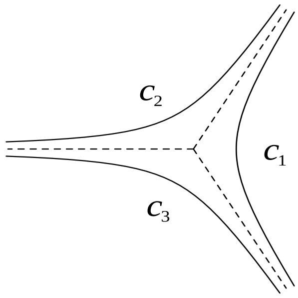

Fig. 1. Cycles for the Airy integral.

Recently, Iwasaki [5] constructed a duality between a pair of polynomial twisted de Rham cohomology groups associated with an isolated surface singularity. As a special case, this construction yields a duality between a pair of twisted de Rham cohomology groups of a generalized Airy function. The aim of this paper is to describe this duality explicitly. Our main result is a formula expressing the intersection matrix relative to Kimura's bases in terms of skew-Schur polynomials (see Theorem 2). This provides us with a cohomological interpretation of skew-Schur polynomials by means of a twisted intersection theory.

2. Generalized Airy function. We begin by recalling the definition of a generalized Airy function, following Gel'fand, Retakh and Serganova [4] and Kimura [7], but in a slightly modified manner. Let $\theta_k(t)$ be the k-th coefficient of a generating series:

$$\log(1 + t_1X + \cdots + t_nX^n) = \sum_{k=1}^{\infty} \theta_k(t)\, X^k.$$

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Table I. Polynomials $\theta_k(t)$

$$\begin{aligned}
\theta_1(t) &= t_1 \\
\theta_2(t) &= t_2 - (1/2)t_1^2 \\
\theta_3(t) &= t_3 - t_2t_1 + (1/3)t_1^3 \\
\theta_4(t) &= t_4 - t_3t_1 - (1/2)t_2^2 \\
&\quad + t_2t_1^2 - (1/4)t_1^4 \\
\theta_5(t) &= t_5 - t_4t_1 - t_3t_2 + t_3t_1^2 \\
&\quad + t_2^2t_1 - t_2t_1^3 + (1/5)t_1^5 \\
\theta_6(t) &= t_6 - t_5t_1 - t_4t_2 + t_4t_1^2 \\
&\quad - (1/2)t_3^2 + 2t_3t_2t_1 - t_3t_1^3 + (1/3)t_2^3 \\
&\quad - (3/2)t_2^2t_1^2 + t_2t_1^4 - (1/6)t_1^6 \\
\theta_7(t) &= \quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Then $\theta_k(t)$ is a weighted homogeneous polynomial of degree k in $t = (t_1, \ldots, t_n)$, where t_j is assumed to be of degree j. For small values of k, polynomials $\theta_k(t)$ are illustrated in Table I. Set

$$f = f(a,t) = \sum_{k=0}^{N} (-1)^k\, e_k(a)\, \theta_{N-k+1}(t),$$

where $e_k(a)$ is the k-th elementary symmetric polynomial of $a = (a_1, \ldots, a_N)$ with $N \geq n$.

Let $\Omega_T^{\cdot}$ be the space of polynomial differential forms in $t = (t_1, \ldots, t_n)$, and $d_{\pm f}$ be the twisted exterior differentials in t defined by

$$d_{\pm f} = e^{\mp f}\, de^{\pm f} = d \pm (df) \wedge .$$

Then one can speak of the twisted polynomial de Rham complexes $(\Omega_T^{\cdot}, d_{\pm f})$ and their cohomology groups $H^{\cdot}(\Omega_T^{\cdot}, d_{\pm f})$. Kimura [7] showed that only the n-th cohomology groups $H^n(\Omega_T^{\cdot}, d_{\pm f})$ are nontrivial with

$$\dim H^n(\Omega_T^{\cdot}, d_{\pm f}) = \mu, \quad \text{where} \quad \mu = \binom{N}{n}.$$

We proceed to homology. Let $T = \mathbf{C}^n$ be the complex n-space with coordinates $t = (t_1, \ldots, t_n)$. Following Pham [13, 14], we define a family Φ of supports in the following manner: an element of Φ is a closed subset c of T such that $\mathrm{Re}\,\theta_{N+1}(t)|_c \to -\infty$, quicker than $-\|t\|^q$ for some $q > 0$ as $\|t\| := \sum_{j=1}^{n} |t_j|^{1/j} \to \infty$. Let $H_n^{\Phi}(T)$ denote the n-th homology group of T over $\mathbf{Z}$ with supports in Φ, which is a free Abelian group of rank μ (see Kimura [8]).

A generalized Airy function is now defined by

$$A(a) = \int_c e^{f(a,t)}\, \omega,$$

where $\omega \in \Omega_T^n$ is a d_f-closed polynomial n-form and c is an n-cycle with support in Φ. This integral depends only on the cohomology class $[\omega] \in H^n(\Omega_T^{\cdot}, d_f)$ and the homology class $[c] \in H_n^{\Phi}(T)$.

3. Bases of cohomology. We present bases of the cohomology groups $H^n(\Omega_T^{\cdot}, d_{\pm f})$. To do so, it is convenient to introduce variables $z = (z_1, \ldots, z_n)$ such that

$$t_j = (-1)^j e_j(z) \quad (j = 1, \ldots, n),$$

where $e_j(z)$ is the j-th elementary symmetric polynomial of z. Given a Young diagram λ, let $s_\lambda(z)$ denote the Schur polynomial in z attached to the diagram λ (see Macdonald [10]). Note that $s_\lambda(z)$ is representable as a polynomial in t.

Let $R(p,q)$ be the rectangular Young diagram with p rows and q columns, and let $\mathcal{Y}(p,q)$ be the set of all Young subdiagrams of $R(p,q)$. The following theorem asserts that the cohomology groups have bases indexed by the Young diagrams in $\mathcal{Y}(n, N-n)$.

Theorem 1. *Denote by $\phi_\lambda^{\pm}$ the cohomology classes in $H^n(\Omega_T^{\cdot}, d_{\pm f})$ represented by the polynomial differential n-form $s_\lambda(z)\,dt$, where $dt = dt_1 \wedge \cdots \wedge dt_n$. Then the sets $\{\phi_\lambda^{\pm} : \lambda \in \mathcal{Y}(n, N-n)\}$ form bases of the cohomology groups $H^n(\Omega_T^{\cdot}, d_{\pm f})$.*

The proof is based on the observation that the graduations $\mathrm{gr}\, H^n(\Omega_T^{\cdot}, d_{\pm f})$ of $H^n(\Omega_T^{\cdot}, d_{\pm f})$ with respect to the degree filtration are linearly isomorphic to the cohomology ring $H^*(\mathrm{Gr}_n(\mathbf{C}^N))$ of the Grassmannian manifold $\mathrm{Gr}_n(\mathbf{C}^N)$ of n-dimensional subspaces in $\mathbf{C}^N$. Then the theorem follows from a standard fact in Schubert calculus (see e.g., Fulton and Pragacz [3, p. 27]). We omit the details.

4. Duality. A general construction in Iwasaki [5] implies that there exists a natural duality between $H^n(\Omega_T^{\cdot}, d_f)$ and $H^n(\Omega_T^{\cdot}, d_{-f})$. In what follows we will briefly describe this duality.

Let $Z = \mathbf{C}^n$ be the complex n-space with coordinates $z = (z_1, \ldots, z_n)$. The symmetric group S_n acts on Z by permuting the coordinates, and one has $T = Z/S_n$. Thus considering the de Rham complexes $(\Omega_T^{\cdot}, d_{\pm f})$ downstairs is equivalent to considering the de Rham complexes $(\Omega_Z^{\cdot}, d_{\pm f})$ upstairs S_n-

equivariantly, where $\Omega_Z^{\cdot}$ is the space of polynomial differential forms in z. Indeed, the canonical projection $Z \to T$ induces isomorphisms:

$$H^n(\Omega_T^{\cdot}, d_{\pm f}) \xrightarrow{\sim} H^n(\Omega_Z^{\cdot}, d_{\pm f})^{S_n},$$

(the transfer isomorphisms), where $H^n(\Omega_Z^{\cdot}, d_{\pm f})^{S_n}$ denote the S_n-invariant parts of $H^n(\Omega_Z^{\cdot}, d_{\pm f})$.

At this stage we should recall a comparison theorem established in Iwasaki [5]. Let $\mathcal{S}_Z^{\cdot}$ be the space of smooth differential forms of Schwartz class on Z, and $\mathcal{T}_Z^{\cdot}$ be the space of tempered currents on Z. Then the natural inclusions of complexes:

$$(\Omega_Z^{\cdot}, d_{\pm f},) \hookrightarrow (\mathcal{T}_Z^{\cdot}, d_{\pm f}) \hookleftarrow (\mathcal{S}_Z^{\cdot}, d_{\pm f})$$

induce S_n-equivariant isomorphisms:

$$H^n(\Omega_Z^{\cdot}, d_{\pm f}) \xrightarrow{\sim} H^n(\mathcal{T}_Z^{\cdot}, d_{\pm f}) \xleftarrow{\sim} H^n(\mathcal{S}_Z^{\cdot}, d_{\pm f}).$$

The topological duality between $\mathcal{S}_Z^{\cdot}$ and $\mathcal{T}_Z^{\cdot}$ induces an S_n-equivariant duality between $H^n(\mathcal{S}_Z^{\cdot}, d_f)$ and $H^n(\mathcal{T}_Z^{\cdot}, d_{-f})$. Thus, through the comparison theorem above, one has an S_n-equivariant duality:

$$H^n(\Omega_Z^{\cdot}, d_f) \times H^n(\Omega_Z^{\cdot}, d_{-f}) \to \mathbf{C},$$

which restricts to the S_n-invariant parts. Now the transfer isomorphisms lead to the desired duality:

$$H^n(\Omega_T^{\cdot}, d_f) \times H^n(\Omega_T^{\cdot}, d_{-f}) \to \mathbf{C}.$$

Explicitly, the duality (or the intersection) pairing between $\phi^{\pm} \in H^n(\Omega_Z^{\cdot}, d_{\pm f})$ is given by

$$\langle \phi^+, \phi^- \rangle = \frac{1}{(2\pi i)^n} \int_Z \psi^+ \wedge \phi^-, \tag{1}$$

where ψ^+ is a smooth differential n-form of Schwartz class on Z corresponding to the cohomology class $\phi^+ \in H^n(\Omega_Z^{\cdot}, d_f)$ through the isomorphism $H^n(\mathcal{S}_Z^{\cdot}, d_f) \xrightarrow{\sim} H^n(\Omega_Z^{\cdot}, d_f)$.

5. Intersection matrix. We will explicitly calculate the intersection matrix relative to the bases constructed in Theorem 1 in terms of skew-Schur polynomials. Let $s_{\lambda/\mu}(a)$ denote the skew-Schur polynomial of $a = (a_1, \ldots, a_N)$ attached to a pair (λ, μ) of Young diagrams (see Macdonald [10]). Note that $s_{\lambda/\mu}(a) \equiv 0$ unless μ is a subdiagram of λ. To state the result, we need the concept of complementary diagrams. Given a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathcal{Y}(p, q)$, its complementary diagram $\check{\lambda} \in \mathcal{Y}(p, q)$ is defined by

$$\check{\lambda} = (q - \lambda_p, q - \lambda_{p-1}, \ldots, q - \lambda_1).$$

Pictorially, $\check{\lambda}$ is obtained by rotating the rectangle $R(p, q)$, together with λ, around its center by 180° and then deleting λ from $R(p, q)$. For instance, a Young diagram $\lambda = (5, 3, 3, 3, 2, 1, 1)$ has the complementary diagram $\check{\lambda} = (4, 4, 3, 2, 2, 2, 0)$ in $\mathcal{Y}(7, 5)$ (see Fig. 2). Taking complementary diagrams $\lambda \mapsto \check{\lambda}$ defines an involution on the set $\mathcal{Y}(p, q)$.

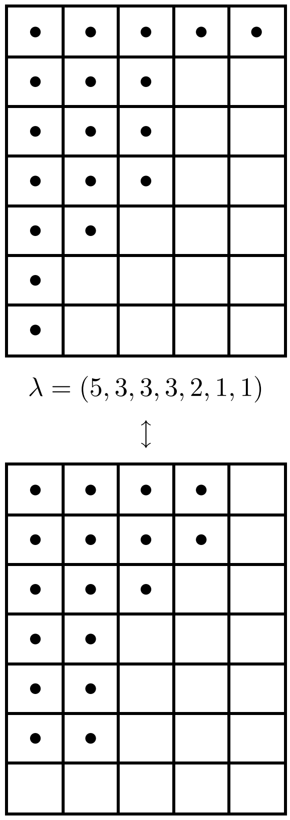

Fig. 2. Complementary diagrams.

Theorem 2. *With respect to the bases* $\{\phi_\lambda^{\pm}\}$ *constructed in* Theorem 1*, the intersection pairing*:

$$H^n(\Omega_T^{\cdot}, d_f) \times H^n(\Omega_T^{\cdot}, d_{-f}) \to \mathbf{C},$$

is represented in the following manner:

$$\langle \phi_\lambda^+, \phi_\mu^- \rangle = (-1)^{n(n-1)/2}\, n!\, s_{\lambda/\check{\mu}}(a),$$

for λ, $\mu \in \mathcal{Y}(n, N-n)$.

We will sketch the proof of Theorem 2 in the following two sections, for the one-dimensional case $n = 1$ in §6, and for the multi-dimensional case in §7, respectively.

6. One-dimensional case. In the one-dimensional case, we have $T = Z = \mathbf{C}$ with coordinate $t = -z$. For $\lambda = (p) \in \mathcal{Y}(1, N-1)$, the cohomology classes $\phi_\lambda^{\pm} \in H^1(\Omega_T^{\cdot}, d_{\pm f})$ are represented by the 1-form $s_\lambda(z)\,dt = -z^p dz$. Since p ranges over the set $\{0,1,\dots,N-1\}$, any elements $\phi^\pm \in H^1(\Omega^{\cdot}_T, d_{\pm f})$ are uniquely expressed as

$$\phi^\pm = -\sum_{k=1}^{N} u_k^\pm z^{N-k}\,dz \quad (u_k^\pm \in \mathbf{C}).$$

Given such a 1-form ϕ^+, there exists a unique formal power series $\psi^+ = \sum_{k=1}^\infty v_k^+ z^{-k} \in \mathbf{C}[[z^{-1}]]$ such that $d_f\psi^+ = \phi^+$. Indeed, this equation is recasted into a recurrence relation for the coefficients v_k^+ that can be solved uniquely. For the first N terms $v_1^+, \dots, v_N^+$, the recurrence relation becomes:

$$\sum_{j=0}^{k-1}(-1)^j e_j(a)\,v_{k-j}^+ = u_k^+ \quad (k=1,\dots,N).$$

Let $h_j(a)$ denote the complete symmetric polynomial of degree j in $a=(a_1,\dots,a_N)$. Then the equations above can be settled as

$$v_k^+ = \sum_{j=0}^{k-1} h_j(a)\,u_{k-j}^+ \quad (k=1,\dots,N).$$

Lemma 3. *With the notations as above, we have*

$$\begin{aligned}\langle\phi^+,\phi^-\rangle &= \mathrm{Res}_{z=\infty}(\psi^+\,\phi^-)\\ &= \sum_{j=1}^{N}\sum_{k=1}^{N} h_{N+1-j-k}(a)\,u_j^+ u_k^-,\end{aligned}$$

where we understand $h_j(a)\equiv 0$ for $j<0$.

Proof. There exists a smooth function ξ^+ on $\mathbf{C}\cup\{\infty\}$ having ψ^+ as its Taylor expansion around $z=\infty$. Then $\varphi^+ := \phi^+ - d_f\xi^+$ becomes a smooth 1-form of Schwartz class that represents the cohomology class ϕ^+ in $H^1(\mathcal{S}^{\cdot}_{\mathbf{C}}, d_f)$. From the integral representation (1) of the duality, one has

$$\begin{aligned}\langle\phi^+,\phi^-\rangle &= \frac{1}{2\pi i}\int_{\mathbf{C}} \varphi^+\wedge\phi^-\\ &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{D_r}\varphi^+\wedge\phi^-,\end{aligned}$$

where D_r is the disk of radius r with center at the origin. Since φ^+ is a 1-form having $-\bar\partial\xi^+$ as its $(0,1)$-component and ϕ^- is a holomorphic 1-form, it follows that $\varphi^+\wedge\phi^- = -(\bar\partial\xi^+)\wedge\phi^- = -\bar\partial(\xi^+\phi^-) = -d(\xi^+\phi^-)$. Substituting this into the above and using the Stokes theorem yield

$$\begin{aligned}\langle\phi^+,\phi^-\rangle &= -\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\partial D_r}\xi^+\phi^-\\ &= -\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\partial D_r}(\xi^+-\eta^+)\,\phi^-\\ &\quad -\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\partial D_r}\eta^+\phi^-,\end{aligned}$$

where $\eta^+ := \sum_{k=1}^m v_k^+ z^{-k}$ is a finite partial sum of the formal power series ψ^+. If m is sufficiently large, then the first term in the right-hand side vanishes and the second term is nothing other than the residue of $\eta^+\phi^-$ at $z=\infty$. Thus we have $\langle\phi^+,\phi^-\rangle = \mathrm{Res}_{z=\infty}(\eta^+\phi^-) = \mathrm{Res}_{z=\infty}(\psi^+\phi^-)$, which verifies the first equality of the lemma. The second equality is obtained by substituting the explicit formula for ψ^+ into the first equality. □

With Lemma 3 in hand, it is now easy to prove Theorem 2 for $n=1$. For $\lambda=(p)$, $\mu=(q)\in\mathcal{Y}(1,N-1)$, let $\phi^+=\phi_\lambda^+$ and $\phi^-=\phi_\mu^-$. Then we have $u_j^+=\delta_{j,N-p}$ and $u_k^-=\delta_{k,N-q}$, where δ_{jk} denotes Kronecker's symbol. It follows from Lemma 3 that $\langle\phi_\lambda^+,\phi_\mu^-\rangle = h_{p+q+1-N}(a) = s_{\lambda/\check\mu}(a)$.

7. Multi-dimensional case. The multi-dimensional case is reduced to the one-dimensional case by making use of an exterior power structure of the cohomology groups $H^n(\Omega^{\cdot}_T, d_{\pm f})$, compatible with the intersection pairing. This reduction completely fits in with the Jacobi-Trudi formulas for Schur and skew-Schur polynomials.

Let $W=\mathbf{C}$ be the complex 1-space with coordinate w, and g be a polynomial in w defined by

$$g(w) = -\sum_{k=0}^{N}\frac{(-1)^k\,e_k(a)}{N+1-k}w^{N+1-k}.$$

Then the arguments and results in §6 apply to the cohomology groups $H^1(\Omega^{\cdot}_W, d_{\pm g})$, since g is just the one-dimensional case of f.

Lemma 4. *There exist natural isomorphisms*:

$$\wedge^n H^1(\Omega^{\cdot}_W, d_{\pm g}) \xrightarrow{\sim} H^n(\Omega^{\cdot}_Z, d_{\pm f})^{S_n}, \tag{2}$$

such that if $\xi_1^\pm\wedge\cdots\wedge\xi_n^\pm$ on the left-hand side correspond to $\varphi^\pm$ on the right-hand side, then

$$n!\det(\langle\xi_j^+,\xi_k^-\rangle) = \langle\varphi^+,\varphi^-\rangle. \tag{3}$$

Proof. Let $Z_j=\mathbf{C}$ be the complex 1-space with coordinate z_j, and set $f_j=g(z_j)$ for $j=1,\dots,n$. Note that each $(\Omega^{\cdot}_{Z_j}, d_{\pm f_j})$ is a copy of $(\Omega^{\cdot}_W, d_{\pm g})$. A key observation here is that we have $f = f_1 + \cdots + f_n$. Hence there are natural isomorphisms of complexes: $\bigotimes_{j=1}^n (\Omega^{\cdot}_{Z_j}, d_{\pm f_j}) \xrightarrow{\sim} (\Omega^{\cdot}_Z, d_{\pm f})$. So the Künneth formula yields isomorphisms:

$$\text{(4)} \qquad \bigotimes_{j=1}^{n} H^1(\Omega^{\cdot}_{Z_j}, d_{\pm f_j}) \xrightarrow{\sim} H^n(\Omega^{\cdot}_Z, d_{\pm f}),$$

sending $\varphi_1^{\pm} \otimes \cdots \otimes \varphi_n^{\pm}$ to $p_1^* \varphi_1^{\pm} \wedge \cdots \wedge p_n^* \varphi_n^{\pm}$, where $p_j : Z = Z_1 \times \cdots \times Z_n \to Z_j$ is the projection down to the j-th component Z_j of Z. The same arguments apply to the complexes $(\mathcal{S}^{\cdot}_{Z_j}, d_{\pm f_j})$ and $(\mathcal{S}^{\cdot}_Z, d_{\pm f})$, provided that the tensor products are understood to be topological tensor products on nuclear spaces. Thus one has

$$\bigotimes_{j=1}^{n} H^1(\mathcal{S}^{\cdot}_{Z_j}, d_{\pm f_j}) \xrightarrow{\sim} H^n(\mathcal{S}^{\cdot}_Z, d_{\pm f}).$$

From these observations it is easy to see that if $\varphi_1^{\pm} \otimes \cdots \otimes \varphi_n^{\pm}$ correspond to $\varphi^{\pm}$ under the isomorphisms (4), then one has

$$\text{(5)} \qquad \prod_{j=1}^{n} \langle \varphi_j^+, \varphi_j^- \rangle = \langle \varphi^+, \varphi^- \rangle.$$

Let $\pi_j : Z_j \to W$ be the canonical isomorphism defined by the change of coordinates: $z_j \mapsto w$. Then there exist well-defined homomorphisms:

$$\text{(6)} \qquad \wedge^n H^1(\Omega^{\cdot}_W, d_{\pm g}) \to \bigotimes_{j=1}^{n} H^1(\Omega^{\cdot}_{Z_j}, d_{\pm f_j}),$$

sending $\xi_1^{\pm} \wedge \cdots \wedge \xi_n^{\pm} \in \wedge^n H^1(\Omega^{\cdot}_W, d_{\pm g})$ to

$$\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \pi_1^* \xi_{\sigma(1)}^{\pm} \otimes \cdots \otimes \pi_n^* \xi_{\sigma(n)}^{\pm}.$$

An inspection shows that the composite of (4) with (6) yields isomorphisms (2) sending $\xi_1^{\pm} \wedge \cdots \wedge \xi_n^{\pm}$ to

$$\varphi^{\pm} = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) p_1^* \pi_1^* \xi_{\sigma(1)}^{\pm} \wedge \cdots \wedge p_n^* \pi_n^* \xi_{\sigma(n)}^{\pm}.$$

Finally, one can easily verify the formula (3) by using (5). The lemma is established. □

We are now in a position to prove Theorem 2 for a general n. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{Y}(n, N-n)$, and set $\xi_j^+ = -w^{\lambda_j + n - j}\, dw$ and $\xi_k^- = -w^{\mu_k + n - k}\, dw$ for $j = 1, \ldots, n$. Then it follows from the calculation in §6 that

$$\begin{aligned} \langle \xi_j^+, \xi_k^- \rangle &= h_{(\lambda_j + n - j) + (\mu_k + n - k) + 1 - N}(a) \\ &= h_{\lambda_j - \check{\mu}_{n+1-k} - j + (n+1-k)}(a), \end{aligned}$$

where the relation $\mu_k = N - n - \check{\mu}_{n+1-k}$ is used in the second equality. Therefore,

$$\begin{aligned} \det(\langle \xi_j^+, \xi_k^- \rangle) &= \det(h_{\lambda_j - \check{\mu}_{n+1-k} - j + (n+1-k)}(a)) \\ &= (-1)^{n(n-1)/2} \det(h_{\lambda_j - \check{\mu}_k - j + k}(a)). \end{aligned}$$

By the Jacobi-Trudi formula for skew-Schur polynomials (see e.g., Macdonald [10]), one has

$$\text{(7)} \qquad \det(\langle \xi_j^+, \xi_k^- \rangle) = (-1)^{n(n-1)/2} s_{\lambda/\check{\mu}}(a).$$

On the other hand, the isomorphism (2) maps $\xi_1^+ \wedge \cdots \wedge \xi_n^+$ to

$$\begin{aligned} \varphi^+ &= (-1)^n \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) z_1^{\lambda_{\sigma(1)} + n - \sigma(1)} dz_1 \wedge \cdots \\ &\qquad \cdots \wedge z_n^{\lambda_{\sigma(n)} + n - \sigma(n)} dz_n \\ &= (-1)^n \det(z_j^{\lambda_k + n - k})\, dz_1 \wedge \cdots \wedge dz_n \\ &= (-1)^n \frac{\det(z_j^{\lambda_k + n - k})}{\det(z_j^{n-k})}\, dt_1 \wedge \cdots \wedge dt_n. \end{aligned}$$

Similarly, (2) maps $\xi_1^- \wedge \cdots \wedge \xi_n^-$ to

$$\varphi^- = (-1)^n \frac{\det(z_j^{\mu_k + n - k})}{\det(z_j^{n-k})}\, dt_1 \wedge \cdots \wedge dt_n.$$

By the Jacobi-Trudi formula for Schur polynomials (see e.g., Macdonald [10]), one has

$$\text{(8)} \qquad \begin{cases} \varphi^+ = (-1)^n s_\lambda(z)\, dt = (-1)^n \phi_\lambda^+, \\ \varphi^- = (-1)^n s_\mu(z)\, dt = (-1)^n \phi_\mu^-. \end{cases}$$

Substitution of (7) and (8) into (3) yields the formula in Theorem 2. The proof is complete. □

References

[1] Airy, G. B.: On the intensity of light in the neighborhood of a caustic. Trans. Cambridge Phil. Soc., **6**, 379–402 (1838).

[2] Aomoto, K., and Kita, M.: Hypergeometric functions. Springer, Tokyo (1994).

[3] Fulton, W., and Pragacz, P.: Schubert varieties and degeneracy loci. Lecture Notes in Math., **1689**, Springer, Berlin (1998).

[4] Gel'fand, I. M., Retakh, V. S., and Serganova, V. V.: Generalized Airy functions, Schubert cells and Jordan groups. Dokl. Akad. Nauk SSSR, **298**, 17–21 (1988); Soviet Math. Dokl., **37**, 8–12 (1988) (English transl.).

[5] Iwasaki, K.: Isolated singularity, Witten's Laplacian, and duality for twisted de Rham cohomology (2000) (preprint).

[6] Iwasaki, K., and Kita, M.: Exterior power structure on the twisted de Rham cohomology of the

complements of real Veronese arrangements. J. Math. Pures Appl., **75**, 69–84 (1995).

[7] Kimura, H.: On rational de Rham cohomology associated with generalized Airy functions. Ann. Scuola Norm. Sup. Pisa, Cl. Sci., (4) **24**, 351–366 (1997).

[8] Kimura, H.: On the homology group associated with the general Airy integral. Kumamoto J. Math., **10**, 11–29 (1997).

[9] Kita, M., and Yoshida, M.: Intersection theory for twisted cycles I, II. Math. Nachr., **166**, 287–304 (1994); **168**, 171–190 (1994).

[10] Macdonald, I. G.: Symmetric functions and Hall polynomials. 2nd ed., Clarendon Press, Oxford (1995).

[11] Matsumoto, K.: Intersection numbers for 1-forms associated with confluent hypergeometric functions. Funkcial. Ekvac., **41**, 291–308 (1998).

[12] Matsumoto, K., and Yoshida, M.: Recent progress of intersection theory for twisted (co)homology groups. (eds. Falk, M. J., and Terao, H.) Arrangements: Tokyo 1998, Adv. Stud. Pure Math., Kinokuniya, Tokyo 1998, pp. 217–237 (2000).

[13] Pham, F.: Vanishing homologies and the n-variable saddle point method. Proc. Symp. Pure Math., **40**, 319–333 (1983).

[14] Pham, F.: La descente des cols par les onglets de Lefschetz avec vues sur Gauss-Manin. Astérisque, **130**, 11–47 (1985).

[15] Yoshida, M.: Intersection theory for twisted cycles III–Determinant formulae, Math. Nachr., **214**, 173–185 (2000).