An analogue of Yi's theorem to holomorphic mappings

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Abstract: This paper gives pairs of explicit hypersurfaces (S_1, S_2) of each complex projective space $\boldsymbol{P}$ for which holds an analogue of Yi's uniqueness theorem [Y]: two linearly nondegenerate holomorphic mappings $f, g\colon \boldsymbol{C} \to \boldsymbol{P}$ are equal if $f^{-1}(S_j) = g^{-1}(S_j)$ $(j = 1, 2)$ as divisors.

Key words: Uniqueness theorem; Nevanlinna theory.

1. Introduction. In [Y], Yi gave some answers for Gross' problem of uniqueness of entire functions with the same inverse images of two finite sets counting multiplicities. Also, he gave its analogue to meromorphic functions with the same inverse images of two finite sets and pole counting multiplicities. It is not difficult to raise a problem whether there exist two hypersurfaces in the complex projective space such that two holomorphic mappings of $\boldsymbol{C}$ into the complex projective space which have the same inverse images of two given hypersurfaces as divisors are identical.

In this paper, we give such hypersurfaces, what we call, of Fermat type.

2. Previous results. Let S be a set of $\overline{\boldsymbol{C}} := \boldsymbol{C} \cup \{\infty\}$. We say that two nonconstant meromorphic functions f and g share S CM(counting multiplicities), if $f^{-1}(S) = g^{-1}(S)$ and the multiplicities of zero of $f - f(z_0)$ and $g - g(z_0)$ at z_0 are equal for each $z_0 \in f^{-1}(S)$.

Let $w = \exp 2\pi i/n$ and $u = \exp 2\pi i/m$ for positive integers n and m. Yi gave the following theorem as an answer for Gross' problem in [Y]:

Theorem A. *Let* $S_1 = \{a_1 + b_1, a_1 + b_1 w, \dots, a_1 + b_1 w^{n-1}\}$, $S_2 = \{a_2 + b_2, a_2 + b_2 u, \dots, a_2 + b_2 u^{m-1}\}$, *where* $n > 4, m > 4,$ a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1 b_2 \neq 0$ *and* $a_1 \neq a_2$. *Suppose that* f *and* g *are nonconstant entire functions sharing* S_1 *and* S_2 *CM. Then* $f = g$.

Also, he gave its analogue to meromorphic functions:

Theorem B. *Let* $S_1 = \{a_1 + b_1, a_1 + b_1 w, \dots, a_1 + b_1 w^{n-1}\}$, $S_2 = \{a_2 + b_2, a_2 + b_2 u, \dots, a_2 + b_2 u^{m-1}\}$, *where* $n > 6, m > 6,$ a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1 b_2 \neq 0$ *and* $a_1 \neq a_2$. *Suppose that* f *and* g *are nonconstant meromorphic functions sharing* S_1, S_2 *and* $\{\infty\}$ *CM. Then* $f = g$.

The aim of this paper is to give an analogue of Theorem A to holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ without third set. Note that the points of S_1 are zero points of $(z - a_1)^n - {b_1}^n$. Let $z = z_1/z_0$, then S_1 is the zero set of $(z_1 - a_1 z_0)^n - (b_1 z_0)^n$ of $\boldsymbol{P}^1(\boldsymbol{C})$, which is a Fermat hypersurface.

In the last of this section we give a useful theorem by Green and Fujimoto and some definitions. We mean by a nonzero entire function an entire function with a point whose value is not zero. For two nonzero entire functions f and g, we say that they are equivalent if the ratio f/g is constant. This introduces an equivalence relation in each set of nonzero entire functions. The following theorem was given in [G] and [F]:

Theorem C. *Let* $f_0, \dots, f_n$ *be nonzero entire functions such that* ${f_0}^d + \cdots + {f_n}^d = 0$, *where* d *is a positive integer. If* $d \geq n^2$, *then*

$$\sum_{f_j \in I} {f_j}^d = 0$$

for each equivalence class I. *Especially each class has at least two elements.*

Definition 1. Let f be a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$. A representation $\tilde{f} = (f_0, \dots, f_n)$ of f is a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{C}^{n+1}$ such that $\tilde{f}^{-1}(\boldsymbol{0}) \neq \boldsymbol{C}$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ for each $z \in \boldsymbol{C} \setminus \tilde{f}^{-1}(\boldsymbol{0})$. A representation $\tilde{f}$ is called to be reduced if $\tilde{f}^{-1}(\boldsymbol{0}) = \emptyset$.

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Definition 2. A holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ is linearly non-degenerate if its image is not contained in any hyperplane of $\boldsymbol{P}^n(\boldsymbol{C})$. This is equivalent to that $f_0, \ldots, f_n$ are linearly independent over $\boldsymbol{C}$, where $(f_0, \ldots, f_n)$ is a representation of f.

3. Uniqueness of holomorphic mappings. Fix a homogeneous coordinate system $(w_0 : \cdots : w_n)$ of $\boldsymbol{P}^n(\boldsymbol{C})$ and consider the Fermat hypersurface S_1 defined by $P_1(w_0, \ldots, w_n) := {w_0}^{p_1} + \cdots + {w_n}^{p_1} = 0$, where p_1 is a positive integer. Let $A = (a_{jk})_{0 \leq j,k \leq n} \in GL(n+1, \boldsymbol{C})$ and consider another Fermat hypersurface S_2 defined by

$$P_2(w_0, \ldots, w_n) := \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_{jk} w_k \right)^{p_2} = 0,$$

where p_2 is a positive integer.

Then S_1 and S_2 give our analogue of Theorem A:

Theorem. *Let f and g be linearly non-degenerate holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with reduced representations $\tilde{f} = (f_0, \ldots, f_n)$ and $\tilde{g} = (g_0, \ldots, g_n)$, respectively. Assume that $p_1, p_2 \geq (2n+1)^2$ and that*

$(*)$ $\quad (a_{jk})^{2p(n+1)} \neq (a_{\mu\nu})^{2p(n+1)}$ *for any (j,k) and (μ,ν) such that $(j,k) \neq (\mu,\nu)$*

where p is the least common multiple of p_1 and p_2. If

$$P_1(f_0, \ldots, f_n) = \alpha^{p_1} P_1(g_0, \ldots, g_n) \tag{1}$$

and

$$P_2(f_0, \ldots, f_n) = \beta^{p_2} P_2(g_0, \ldots, g_n) \tag{2}$$

hold for some entire functions α and β without zeros, i.e., $f^*(S_j) = g^*(S_j)$ $(j = 1, 2)$ as divisors, then $f = g$.

Proof. At the beginning of proof, we note that none of f_j, g_j, $\sum_{k=0}^{n} a_{jk} f_k$, $\sum_{k=0}^{n} a_{jk} g_k$ is identically equal to zero by linear non-degeneracy of f and g. We apply Theorem C to (1) considering linear non-degeneracy of f and g. Then there exist a permutation σ_j of $0, \ldots, n$ and p_1-th roots ω_j of 1 such that

$$f_j = \omega_j \alpha g_{\sigma_j} \quad (0 \leq j \leq n). \tag{3}$$

Similarly, from (2) we have a permutation τ_j of $0, \ldots, n$ and p_2-th roots η_j of 1 such that

$$\sum_{k=0}^{n} a_{jk} f_k = \eta_j \beta \sum_{k=0}^{n} a_{\sigma_j k} g_k \quad (0 \leq j \leq n). \tag{4}$$

We represent (3) and (4) by matrices:

$$ {}^t\tilde{f} = \alpha \Omega R\, {}^t\tilde{g}, \tag{5}$$

$$ A\, {}^t\tilde{f} = \beta H T A\, {}^t\tilde{g}, \tag{6}$$

where

$$\Omega = \begin{pmatrix} \omega_0 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{pmatrix}, \quad H = \begin{pmatrix} \eta_0 & & 0 \\ & \ddots & \\ 0 & & \eta_n \end{pmatrix}$$

and $R = \left(\delta_{\sigma_j k} \right)_{0 \leq j,k \leq n}$, $T = \left(\delta_{\tau_j k} \right)_{0 \leq j,k \leq n}$, where δ_{jk} is Kronecker's delta. Now, by deleting ${}^t\tilde{f}$ from (5) and (6) we get

$$(\alpha A \Omega R - \beta H T A)\, {}^t\tilde{g} = {}^t(0, \ldots, 0). \tag{7}$$

Since $\det(\alpha A \Omega R - \beta H T A) \equiv 0$ trivially, $\det\{(\alpha/\beta)E - (A\Omega R)^{-1}(HTA)\} \equiv 0$, where E is the identity matrix of size $n+1$. This means that the function α/β takes value in the set of the eigenvalues of $(A\Omega R)^{-1}(HTA)$, and hence, $\alpha/\beta = c$, where c is a nonzero constant. By substituting $\alpha = c\beta$ into (7), we have

$$\beta(cA\Omega R - HTA)\, {}^t\tilde{g} = {}^t(0, \ldots, 0).$$

Again, linear non-degeneracy of g induces

$$cA\Omega R = HTA, \tag{8}$$

By taking determinants of both sides, $c^{n+1} \omega_0 \cdots \omega_n \times \det R = \eta_0 \cdots \eta_n \det T$ is obtained, and hence $c^{2p(n+1)} = 1$. From (8), $cH^{-1}A\Omega = TAR^{-1} = TA\, {}^tR$ and by comparing each coefficients $c a_{jk} \omega_k / \eta_j = a_{\sigma_j \tau_k}$. It follows from the condition $(*)$ that $\sigma_j = j$, $\tau_k = k$. As $R = T = E$, we get $c\omega_0 = \cdots = c\omega_n = \eta_0 \cdots = \eta_n$ from (8). It implies $f = g$ by (5) or (6). □

Now, we apply this theorem to the uniqueness of linearly non-degenerate holomophic mappings by one hypersurface as in [S1] and [S2]. Let $P(w_0, w_1)$ be a homogeneous polynomial of degree d with the following property:

(U1) let f and g be nonconstant holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with reduced representations $\tilde{f} = (f_0, f_1)$ and $\tilde{g} = (g_0, g_1)$, respectively. If $P(f_0, f_1) = \alpha^d P(g_0, g_1)$ holds for an entire function α without zeros, then $f_j = \omega \alpha g_j \quad (j = 0, 1)$, where $\omega^d = 1$.

The existence of such polynomial is shown in [S1], where the least degree is 13.

From Theorem A we can prove easily

Corollary. *In the above situation, assume that* $p_1 = p_2 \geq (2n+1)^2$ *and that the condition* $(*)$ *in Theorem is satisfied. Let* S *be a hypersurface in* $\boldsymbol{P}^n(\boldsymbol{C})$ *defined by*

$$P(P_1(w_0, \ldots, w_n), P_2(w_0, \ldots, w_n)) = 0.$$

Then two linearly non-degenerate holomorphic mappings f *and* g *of* $\boldsymbol{C}$ *into* $\boldsymbol{P}^n(\boldsymbol{C})$ *satisfying* $f^*(S) = g^*(S)$ *as divisors are identical.*

In [S2], the author gave another hypersurface of degree $d(2n-1)^2$ which is smaller than $d(2n+1)^2$ of the least degree of hypersurfaces in Corollary.

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