Bernstein degree of singular unitary highest weight representations of the metaplectic group

By Kyo Nishiyama*) and Hiroyuki Ochiai**)

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Let ω be the Weil representation of the metaplectic double cover $G = Mp(2n, \mathbf{R})$ of the symplectic group $Sp(2n, \mathbf{R})$ of rank n. Consider the m-fold tensor product $\omega^{\otimes m}$ of ω. Then the orthogonal group $O(m)$ acts on $\omega^{\otimes m}$ from the right and the action generates the full algebra of intertwiners. Therefore we can decompose $\omega^{\otimes m}$ as $G \times O(m)$-module (see [6, 7]):

$$\omega^{\otimes m} = \bigoplus_{\sigma \in \widehat{O}(m)} L(\sigma) \otimes \sigma.$$

In this article, we consider $L(\mathbf{1}_m)$ $(1 \le m \le n)$ which corresponds to the trivial representation $\mathbf{1}_m$ of $O(m)$. If $1 \le m \le n$, $L(\mathbf{1}_m)$ is an irreducible singular unitary highest weight representation of G and it has one-dimensional minimal K-type. Note that, if m is even, then $L(\mathbf{1}_m)$ factors through and gives an irreducible representation of $Sp(2n, \mathbf{R})$.

The aim of this article is to give a formula for the Bernstein degree of $L(\mathbf{1}_m)$, which is denoted by $\operatorname{Deg} L(\mathbf{1}_m)$ (See Section 1). Main results are Theorem 1.2 and Corollary 2.3. We prove them by using Gindikin gamma function on a positive Hermitian cone in Section 2. On the other hand, the representation $L(\mathbf{1}_m)$ is realized on the so-called determinantal variety, and the calculation of $\operatorname{Deg} L(\mathbf{1}_m)$ is equivalent to obtaining the degree of the determinantal variety. Its degree is already known as Giambelli's formula and proved by Harris and Tu [4] with the help of Thom-Porteous formula. Therefore our formula gives an alternative proof of the Giambelli's formula. We shall explain it briefly in Section 3.

1. Bernstein degree of $L(\mathbf{1}_m)$. Let K be a maximal compact subgroup of G. Then K is isomorphic to the non-trivial double cover of $U(n)$. K-finite vectors in $\omega^{\otimes m}$ can be identified with $\det^{m/2} \otimes \mathbf{C}[M_{n,m}]$ by the Fock realization of ω, where $M_{n,m}$ denotes the space of $n \times m$ matrices. In this picture, K acts naturally from the left (but with the shift by $\det^{m/2}$) and $O(m)$ acts from the right. By the characterization of $L(\mathbf{1}_m)$, we get

$$L(\mathbf{1}_m)\big|_K \simeq \det^{m/2} \otimes \mathbf{C}[M_{n,m}]^{O(m)}.$$

The following lemma is well-known. See [5, p. 35], for example.

Lemma 1.1. *As a representation of $U(n)$, we have the multiplicity free decomposition*

$$\mathbf{C}[M_{n,m}]^{O(m)} \simeq \bigoplus_{l(\lambda) \le m} \tau_{2\lambda},$$

where τ_μ denotes the irreducible finite dimensional representation of $U(n)$ with the highest weight μ, and the summation is taken over all the partition λ of the non-negative integers of length less than or equal to m.

Using this lemma, we can define a natural K-invariant filtration of $L(\mathbf{1}_m)$ by putting $L(\mathbf{1}_m)_k = \det^{m/2} \otimes \left(\bigoplus_{|\lambda| \le k, l(\lambda) \le m} \tau_{2\lambda} \right)$ $(k \ge 0)$. Let $d = \operatorname{Dim} L(\mathbf{1}_m)$ be the *Gelfand-Kirillov dimension* of $L(\mathbf{1}_m)$ and denote by $\operatorname{Deg} L(\mathbf{1}_m)$ the *Bernstein degree* (see [10] for definition). Then the theory of Hilbert polynomials tells us that, for sufficient large k, $\dim L(\mathbf{1}_m)_k$ is a polynomial in k and the top term is given by

$$\dim L(\mathbf{1}_m)_k = \frac{\operatorname{Deg} L(\mathbf{1}_m)}{d!} k^d + (\text{lower terms in } k).$$

It is easy to see that $d = \operatorname{Dim} L(\mathbf{1}_m) = nm - m(m-1)/2$ (cf. Eq. (1) below).

Theorem 1.2. *The Bernstein degree of $L(\mathbf{1}_m)$ is given by*

$$\operatorname{Deg} L(\mathbf{1}_m) = \frac{2^{d-m} d!}{m! \prod_{i=1}^m (n-i)!} \times \int_{x_i \ge 0, \sum_{i=1}^m x_i \le 1} (x_1 x_2 \cdots x_m)^{n-m} \times \prod_{1 \le i < j \le m} |x_i - x_j| dx_1 dx_2 \cdots dx_m.$$

Remark 1.3. We shall give the exact formula for the integral in the next section.

*) Faculty of Integrated Human Studies, Kyoto University, Yoshida Nihonmatsu-cho, Kyoto 606-8501.
**) Department of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-0053.

Proof . By Weyl's dimension formula, we have

(1) $$\operatorname{Deg} L(\mathbf{1}_m) = \lim_{k\to\infty} \frac{d!}{k^d} \sum_{l(\lambda)\le m, |\lambda|\le k} \prod_{\alpha\in\Delta^+} \frac{\langle 2\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle},$$

where Δ^+ is a positive system of the roots of $U(n)$, and $\rho = \sum_{\alpha\in\Delta^+}\alpha/2$. From this formula, we get the integral. □

2. Integral over the positive Hermitian cones. Let us slightly generalize the integral in Theorem 1.2, and put

(2) $$I(s,m) = \int_{x_i\ge 0, \sum_{i=1}^m x_i\le 1} (x_1x_2\cdots x_m)^s \times \prod_{1\le i<j\le m} |x_i - x_j|^\alpha dx_1dx_2\cdots dx_m.$$

We give the exact formula of the integral in this section. It arises as a natural integral over a positive Hermitian cone.

Let $V = \operatorname{Herm}(m,\mathbf{F})$, the space of Hermitian $m\times m$ matrices over the field $\mathbf{F} = \mathbf{R},\mathbf{C},\mathbf{H}$. Put $N = \dim_{\mathbf{R}} V = m + \frac{\alpha}{2}m(m-1)$, where $\alpha = \dim_{\mathbf{R}}\mathbf{F}$ $(= 1,2,4)$. Denote by Ω the cone of positive definite Hermitian matrices with scalar product $(x,y) = \operatorname{Re}\operatorname{trace} xy^*$.

Lemma 2.1. *For a function φ on the interval $[0,\infty)$ and $\operatorname{Re} s > -1$, we have*

(3) $$\int_\Omega \varphi(\operatorname{trace} y)(\det y)^s dy = \frac{\Gamma_\Omega(s+N/m)}{\Gamma(sm+N)} \int_0^\infty \varphi(t)t^{sm+N-1}dt,$$

where dy is the Euclidean measure on V and $\Gamma_\Omega(s)$ is the Gindikin gamma function of the cone Ω (see [1, Chapter VII]).

Proof . Put

$$h(t,s) = \int_\Omega (\det y)^s \delta(\operatorname{trace} y - t) \quad (t>0),$$

where $\delta(x)$ means the Dirac measure. Then the function $h(t,s)$ is homogeneous in t of degree $sm+N-1$, i.e., $h(\lambda t,s) = \lambda^{sm+N-1}h(t,s)$ $(\lambda>0)$. Therefore we have $h(t,s) = g(s)t^{sm+N-1}$, where $g(s) = h(1,s)$. Take $\varphi(t) = e^{-t}$ in the left hand side of the formula (3). Then we get

$$\int_\Omega e^{-\operatorname{trace} y}(\det y)^s dy = g(s)\int_0^\infty e^{-t}t^{sm+N-1}dt = g(s)\Gamma(sm+N).$$

On the other hand, the left hand side of the above formula is $\Gamma_\Omega(s+N/m)$ by definition. So we get $g(s) = \Gamma_\Omega(s+N/m)/\Gamma(sm+N)$ and we have done. □

Theorem 2.2. *Let $I(s,m)$ be as in* (2). *For* $\operatorname{Re} s > -1$ *and* $\alpha = 1,2,4$, *we have*

$$I(s,m) = \frac{\prod_{j=1}^m \Gamma(j\alpha/2+1)\Gamma(s+1+(j-1)\alpha/2)}{\Gamma(\alpha/2+1)^m\Gamma(sm+N+1)}.$$

Proof . If we denote by $(x_1,\cdots,x_m)$ the eigenvalues of $y\in\Omega$, we get

$$\int_\Omega \varphi(\operatorname{trace} y)(\det y)^s dy = c_0\int_{x_i\ge 0}\varphi(\sum_{i=1}^m x_i)(x_1x_2\cdots x_m)^s \times \prod_{1\le i<j\le m}|x_i-x_j|^\alpha dx_1dx_2\cdots dx_m,$$

for some non-zero constant c_0. If we take $\varphi = \chi_{[0,1]}$ the characteristic function of the interval $[0,1]$, we can calculate the integral modulo the constant c_0:

$$I(s,m) = \frac{1}{c_0}\int_\Omega \chi_{[0,1]}(\operatorname{trace} y)(\det y)^s dy = \frac{1}{c_0}\frac{\Gamma_\Omega(s+N/m)}{\Gamma(sm+N)}\int_0^1 t^{sm+N-1}dt \quad \text{(by Lemma 2.1)} = \frac{1}{c_0}\frac{\Gamma_\Omega(s+N/m)}{\Gamma(sm+N+1)}.$$

Since $c_0\prod|x_i-x_j|^\alpha$ appears as the Jacobian of the integral, we can calculate it as

$$(\sqrt{2\pi})^N = \int_V \exp(-\|y\|^2/2)dy = c_0\int_{\mathbf{R}^m}\exp(-\sum_i x_i^2/2) \times \prod|x_i-x_j|^\alpha dx_1\cdots dx_m = c_0(2\pi)^{m/2}\prod_{j=1}^m\frac{\Gamma(j\alpha/2+1)}{\Gamma(\alpha/2+1)}.$$

The last equality follows from Selberg's integral (see [1, p. 121]). Now the formula above and the product formula for the Gindikin gamma function ([1, Chapter VII], see also [9, p. 585]) proves the theorem. □

As for the integral $I(s,m)$, see [8, Hilfsatz 10] also.

Corollary 2.3.

$$\operatorname{Deg} L(\mathbf{1}_m) = \frac{2^d}{\pi^{m/2} m!} \prod_{j=1}^{m} \frac{\Gamma(j/2+1)\Gamma(d/m-(j-1)/2)}{\Gamma(n-j+1)}.$$

3. Degree of the determinantal varieties. Let $\mathrm{Sym}_n(m) = \{X \in M_n(\mathbf{C}) \mid {}^tX = X, \operatorname{rank} X \leq m\}$ be the variety of symmetric matrices of rank at most m. This is called the *determinantal variety.*

Theorem 3.1. *There is a GL_n-equivariant isomorphism $\mathbf{C}[\mathrm{Sym}_n(m)] \simeq \mathbf{C}[M_{n,m}]^{O(m)}$ by the pull back of the following map of the base spaces.*

$$M_{n,m} \ni X \longmapsto X^tX \in \mathrm{Sym}_n(m)$$

Proof . This fact is well-known in invariant theory. See [5] for example. □

The variety of symmetric matrices modulo scalars forms a projective space $\mathbf{P}(\mathrm{Sym}_n)$. The homogeneous coordinate ring of the subvariety $\mathbf{P}(\mathrm{Sym}_n(m))$ is $\mathbf{C}[\mathrm{Sym}_n(m)]$. Therefore the calculation of the degree of $\mathbf{C}[M_{n,m}]^{O(m)}$ is equivalent to that of the determinantal variety. The degree of the subvariety $\mathbf{P}(\mathrm{Sym}_n(m))$ is given by the so-called *Giambelli's formula.*

Theorem 3.2 [4, Proposition 12]**.** *Let $r = n - m$. The degree of the subvariety $\mathbf{P}(\mathrm{Sym}_n(m))$ is given by*

$$\deg(\mathbf{P}(\mathrm{Sym}_n(m))) = \prod_{j=0}^{r-1} \frac{\binom{n+j}{r-j}}{\binom{2j+1}{j}}.$$

This theorem goes back to [3]. In [4], Harris and Tu proved the formula in the geometric way. The representation theoretic degree $\operatorname{Deg} L(\mathbf{1}_m)$ coincides with the formula above, and our formula in Corollary 2.3 gives an alternative proof of it.

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