

On abundance theorem for semi log canonical threefolds

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Abstract: Let (X, Δ) be a proper semi log canonical threefold with $K_X + \Delta$ nef. Then $K_X + \Delta$ is semi-ample.

Key word: Abundance Theorem.

1. Introduction. The main purpose of this paper is to announce the abundance theorem for semi log canonical threefolds (for the proof, see [3]). The abundance conjecture is a very important problem in the birational classification of algebraic varieties.

By our method we can reduce the abundance conjecture for semi log canonical pairs to the irreducible case and the finiteness of some groups. This shows that if the log Minimal Model Program (log MMP, for short), the log abundance conjecture, and the finiteness of B-pluricanonical representations (see Section 4) hold for n -folds, then the log abundance conjecture for semi log canonical n -folds is true almost automatically (see Theorem 5.10). But unfortunately the log MMP and the log abundance conjecture are only conjectures for n -folds with $n \geq 4$. So we prove

Theorem 1.1 (Abundance theorem for slc 3-folds). *Let (X, Δ) be a proper semi log canonical threefold with $K_X + \Delta$ nef. Then $K_X + \Delta$ is semi-ample.*

This theorem is a generalization of the log abundance theorem for 3-folds proved by S. Keel, K. Matsuki and J. McKernan (see [12]). According to them, the log abundance theorem for threefolds is considered to be the first step towards a proof of the abundance conjecture in dimension four. We believe that the abundance theorem for semi log canonical threefolds is the second step.

Notation.

- (1) The word *scheme* is used for schemes which are separated and of finite type over \mathbf{C} and the word *variety* stands for a reduced and irreducible scheme. When we say a normal scheme, we mean the disjoint union of irre-

ducible normal schemes.

- (2) We use the notation that (X, Δ) being *Kawamata log terminal*, *divisorial log terminal*, and *log canonical* (frequently abbreviated as klt, dlt, and lc), as in [15] (see also [20]). In the definition in [15, Section 2.3] Δ is not necessarily effective, but in this paper we assume Δ is an effective \mathbf{Q} -divisor.
- (3) The log MMP means the log MMP for \mathbf{Q} -factorial dlt pairs.

2. Definitions and preliminaries. In this section, we make some basic definitions.

Definition 2.1. Let X be a reduced S_2 scheme. We assume that it is pure n -dimensional and normal crossing in codimension 1. Let Δ be an effective \mathbf{Q} -Weil divisor on X (cf. [14, 16.2]) such that $K_X + \Delta$ is \mathbf{Q} -Cartier.

Let $X = \cup X_i$ be a decomposition into irreducible components, and $\mu : X' := \coprod X'_i \rightarrow X = \cup X_i$ the normalization. A \mathbf{Q} -divisor Θ on X' is defined by $K_{X'} + \Theta := \mu^*(K_X + \Delta)$ and a \mathbf{Q} -divisor Θ_i on X'_i by $\Theta_i := \Theta|_{X'_i}$.

We say that (X, Δ) is a *semi log canonical n -fold* (an slc n -fold, for short) if (X'_i, Θ_i) is lc for every i .

We say that (X, Δ) is a *semi divisorial log terminal n -fold* (an sdlc n -fold, for short) if X_i is normal, that is, X'_i is isomorphic to X_i , and (X_i, Θ_i) is dlt for every i .

The notion of semi log canonical pairs was first introduced in [16] for the problem of compactifying the moduli of surfaces. For the further development of this direction, we recommend the readers to see [7].

Remark 2.2.

- (1) The slc in Definition 2.1 is equivalent to the slc in [14, Chapter 12] (see [13, 4.2]).
- (2) If (X, Δ) is an slc n -fold, then X is seminor-

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mal (see [14, 12.2.1.(8)] and [1, Remark 4.7]).

- (3) If (X, Δ) is a dlt n -fold, then $(\lrcorner\Delta\lrcorner, \text{Diff}(\Delta - \lrcorner\Delta\lrcorner))$ is an sdlc $(n-1)$ -fold (see [14, 17.5] and [15, 5.52]).
- (4) Let (X, Δ) be lc. Then $(\lrcorner\Delta\lrcorner, \text{Diff}(\Delta - \lrcorner\Delta\lrcorner))$ is not necessarily slc (see [14, 17.5.2 Example]). Note that [14, (16.9.1)] is not true.

Definition 2.3. Let pairs $(X, \Delta) = \coprod_{i=1}^n (X_i, \Delta_i)$ and $(X', \Delta') = \coprod_{i=1}^n (X'_i, \Delta'_i)$ be normal schemes with \mathbf{Q} -divisors such that $K_X + \Delta$ and $K_{X'} + \Delta'$ are \mathbf{Q} -Cartier \mathbf{Q} -divisors.

We say that a map $f : (X, \Delta) \dashrightarrow (X', \Delta')$ is a B-birational map (resp. morphism) if $f : X \dashrightarrow X'$ is a proper birational map (resp. morphism) and there exists a common resolution $\alpha : T \rightarrow X$, $\beta : T \rightarrow X'$ of $f : X \dashrightarrow X'$ such that $\alpha^*(K_X + \Delta) = \beta^*(K_{X'} + \Delta')$. That is, there exists a permutation σ such that $f_i : X_i \dashrightarrow X'_{\sigma(i)}$ is a proper birational map (resp. morphism) and there exists a common resolution $\alpha_i : T_i \rightarrow X_i$, $\beta_i : T_i \rightarrow X'_{\sigma(i)}$ of f_i such that $\alpha_i^*(K_{X_i} + \Delta_i) = \beta_i^*(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)})$ on T_i for every i . The last condition means that if we write

$$\begin{aligned} K_{T_i} &= \alpha_i^*(K_{X_i} + \Delta_i) + F, \\ K_{T_i} &= \beta_i^*(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)}) + E, \end{aligned}$$

then $F = E$.

If there exists a B-birational map from (X, Δ) to (X', Δ') , we say that (X, Δ) is B-birational equivalent to (X', Δ') . Here the symbol B is the initial of *boundary*.

3. Reduced boundaries of dlt pairs. The following is a reformulation of [14, 12.3.2], which fits better in our arguments. It plays essential roles in the proof of Proposition 5.3, which is a key step of the proof of the main theorem.

Proposition 3.1 (cf. [19, 6.9], [14, 12.3.2]).

Let (X, Θ) be a \mathbf{Q} -factorial dlt n -fold with $n \leq 3$. Let $f : X \rightarrow R$ be a proper surjective morphism onto a normal variety R with connected fibers. Assume that $K_X + \Theta$ is numerically f -trivial. Then one of the following holds:

- (0) $\dim R = 0$.
- (0.1) $\lrcorner\Theta\lrcorner$ is connected.
- (0.2) $\lrcorner\Theta\lrcorner$ has two connected components Δ_1 and Δ_2 and there exists a rational map $v : X \dashrightarrow (V, P)$ onto a \mathbf{Q} -factorial lc $(n-1)$ -fold (V, P) with general fiber \mathbf{P}^1 . Furthermore there exists an irreducible component $D'_i \subset \Delta_i$ such that $v|_{D'_i} : (D'_i, \text{Diff}(\Theta - D'_i)) \dashrightarrow$

(V, P) is a B-birational map for $i = 1, 2$.

- (1) $\dim R \geq 1$.

(1.1) $\lrcorner\Theta\lrcorner \cap f^{-1}(r)$ is connected for every $r \in R$.

(1.2) The number of connected components of $\lrcorner\Theta\lrcorner \cap f^{-1}(r)$ is at most two for every $r \in R$. There exists a rational map $v : X \dashrightarrow (V, P)$ onto a \mathbf{Q} -factorial lc $(n-1)$ -fold (V, P) with general fiber \mathbf{P}^1 . The horizontal part Θ^h of $\lrcorner\Theta\lrcorner$ is one of the following:

- (i) $\Theta^h = D'_1$ which is irreducible and the mapping degree $\deg[D'_1 : V] = 2$ and there is a B-birational involution on $(D'_1, \text{Diff}(\Theta - D'_1))$ over V .
- (ii) $\Theta^h = D'_1 + D'_2$ such that D'_i is irreducible and $v|_{D'_i} : (D'_i, \text{Diff}(\Theta - D'_i)) \dashrightarrow (V, P)$ is a B-birational map for $i = 1, 2$.

Remark 3.2.

- (1) In Proposition 3.1 the assumption $K_X + \Theta \equiv_f 0$ is equivalent to $K_X + \Theta \sim_{\mathbf{Q}, f} 0$. It is because the relative log abundance theorem holds when $\dim X \leq 3$.
- (2) If the log MMP holds for n -folds, then Proposition 3.1 is also true for n -folds.

4. Finiteness of B-pluricanonical representations. We consider the birational automorphism groups of pairs.

Definition 4.1. Let (X, Δ) be a pair of a normal scheme and a \mathbf{Q} -divisor such that $K_X + \Delta$ is \mathbf{Q} -Cartier. We define

$$\begin{aligned} \text{Bir}(X, \Delta) &:= \{ \sigma : X \dashrightarrow X \mid \sigma \text{ is B-birational} \}, \\ \text{Aut}(X, \Delta) &:= \left\{ \sigma : X \rightarrow X \left| \begin{array}{l} \text{The map } \sigma \text{ is an} \\ \text{automorphism and} \\ \sigma^* \Delta = \Delta \end{array} \right. \right\}. \end{aligned}$$

Since $\text{Bir}(X, \Delta)$ acts on $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ for every integer m such that $m(K_X + \Delta)$ is a Cartier divisor, we can define B-pluricanonical representation $\rho_m : \text{Bir}(X, \Delta) \rightarrow \text{Aut } H^0(X, m(K_X + \Delta))$.

The following conjecture plays an important role when we reduce the problem to the irreducible case.

Conjecture 4.2 (Finiteness of B-pluricanonical representations). Let (X, Δ) be an n -dimensional (not necessarily connected) proper lc pair. Assume that $K_X + \Delta$ is nef. Then there exists a positive integer m_0 such that $\rho_{m_1 m_0}(\text{Bir}(X, \Delta))$ is finite for every $m_1 \in \mathbf{N}$.

For Conjecture 4.2, it is obviously sufficient to prove it under the assumption that X is irreducible.

Theorem 4.3 (cf. [14, 12.2.11]). *Let (C, Δ) be a proper lc curve. Then there exists a positive integer m_0 such that $\rho_{m_1 m_0}(\text{Aut}(C, \Delta))$ is finite for every $m_1 \in \mathbf{N}$.*

Theorem 4.4. *Let (S, Δ) be a proper klt surface. Then there exists a positive integer m_0 such that $\rho_{m_1 m_0}(\text{Bir}(S, \Delta))$ is finite for every $m_1 \in \mathbf{N}$.*

Theorem 4.5. *Let (S, Δ) be a proper lc surface. Assume that $K_S + \Delta$ is nef. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\text{Bir}(S, \Delta))$ is finite for every $m_1 \in \mathbf{N}$.*

5. The abundance theorem for slc pairs.

We introduce the notion of preadmissible and admissible sections for the inductive proof of the abundance conjecture for slc n -folds.

Definition 5.1. Let (X, Δ) be a proper sldt n -fold and m a sufficiently large and divisible integer. We define admissible and preadmissible sections inductively on dimension.

- (1) $s \in H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is *preadmissible* if the restriction $s|_{(\coprod_i \perp \Theta_i \lrcorner)}$ is admissible in $H^0(\coprod_i \perp \Theta_i \lrcorner, \mathcal{O}_{(\coprod_i \perp \Theta_i \lrcorner)}(m(K_{X'} + \Theta)|_{(\coprod_i \perp \Theta_i \lrcorner)}))$.
- (2) $s \in H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is *admissible* if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every B-birational map $g : (X_i, \Theta_i) \dashrightarrow (X_j, \Theta_j)$ for every i, j .

Note that if $s \in H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is admissible, then $s|_{X_i}$ is $\text{Bir}(X_i, \Theta_i)$ -invariant for every i .

We define linear subspaces of $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ as follows;

$$\text{PA}(X, m(K_X + \Delta)) := \{s \text{ is preadmissible}\},$$

and

$$\text{A}(X, m(K_X + \Delta)) := \{s \text{ is admissible}\}.$$

When $\dim X = 1$, the preadmissible section is a slight generalization of the *normalized* section (see [14, 12.2.9]). But in the higher dimensional case, the admissible and preadmissible sections behave much better in the inductive proof of the abundance conjecture for slc n -folds.

Lemma 5.2. *Let (X, Δ) be a proper \mathbf{Q} -factorial dlt n -fold, and $K_X + \Delta$ nef and $S = \perp \Delta \lrcorner \neq 0$. Assume that $f = \Phi_{|k(K_X + \Delta)|} : X \rightarrow R$ is a proper surjective morphism onto a normal variety R with connected fibers for a sufficiently large and divisible integer k and $f(\perp \Delta \lrcorner) \subsetneq R$. If there exist sections $\{s_i\}_{i=1}^p \subset H^0(S, \mathcal{O}_S(m(K_X + \Delta)|_S))$ without*

common zeros, then there exist sections $\{u_i\}_{i=1}^l \subset H^0(X, \mathcal{O}_X(\text{rm}(K_X + \Delta)))$ for some integer r such that

$$(1) \quad u_i|_S = \begin{cases} s_i^r & \text{for } 1 \leq i \leq p \\ 0 & \text{for } p < i \leq l \end{cases}$$

$$(2) \quad \{u_i\}_{i=1}^l \text{ have no common zeros.}$$

The next proposition is the main part of the proof of the abundance theorem for slc n -folds, which is proved by Proposition 3.1 and Lemma 5.2.

Proposition 5.3. *Let (X, Δ) be a pure n -dimensional (not necessarily connected) proper \mathbf{Q} -factorial dlt pair with $n \leq 3$. Let m be a sufficiently large and divisible integer. Assume that*

- (1) $K_X + \Delta$ is nef,
- (2) $\mathcal{O}_{\perp \Delta \lrcorner}(m(K_{\perp \Delta \lrcorner} + \text{Diff}(\Delta - \perp \Delta \lrcorner)))$ is generated by $\text{A}(\perp \Delta \lrcorner, m(K_X + \Delta)|_{\perp \Delta \lrcorner})$.

Then

$$\text{PA}(X, m(K_X + \Delta)) \rightarrow \text{A}(\perp \Delta \lrcorner, m(K_X + \Delta)|_{\perp \Delta \lrcorner})$$

is surjective, and $\mathcal{O}_X(m(K_X + \Delta))$ is generated by $\text{PA}(X, m(K_X + \Delta))$.

Remark 5.4. In Proposition 5.3 the assumption $n \leq 3$ is used for the log MMP and the log abundance theorem which are used in Proposition 3.1 too.

By Lemma 5.7 and Section 4, we get the next lemma.

Lemma 5.5. *In Proposition 5.3, if $\dim X \leq 2$, then we can replace $\text{PA}(X, m(K_X + \Delta))$ by the linear subspace $\text{A}(X, m(K_X + \Delta))$.*

In order to prove Lemma 5.7 we need the next definition.

Definition 5.6. Assume that X is nonsingular and $\text{Supp } \Delta$ is a simple normal crossing divisor and $\Delta = \sum_i d_i \Delta_i$ is a \mathbf{Q} -divisor such that $d_i \leq 1$ for every i . In this case we say that (X, Δ) is *B-smooth*.

Let (X, Δ) be dlt or B-smooth. A subvariety W of X is said to be a *center of log canonical singularities* if there is a proper birational morphism from a normal variety $\mu : Y \rightarrow X$ and a prime divisor E on Y with the discrepancy $a(E, X, \Delta) = -1$ such that $\mu(E) = W$ (cf. [10, Definition 1.3]).

Let (X, Δ) be dlt or B-smooth. We write $\Delta = \sum_i d_i \Delta_i$ such that Δ_i are distinct prime divisors. Then the *B-part* of Δ is defined by $\Delta^B := \sum_{d_i=1} \Delta_i$.

If (X, Δ) is dlt or B-smooth, then a center of log canonical singularities is an irreducible component of an intersection of some B-part divisors. (see the Divisorial Log Terminal Theorem of [20] and [15, Sec-

tion 2.3].) When we consider a center of log canonical singularities W , we always consider the pair (W, Ξ) such that $K_W + \Xi = (K_X + \Delta)|_W$, where Ξ is defined by using the adjunction repeatedly. Note that if (X, Δ) is dlt (resp. B-smooth), then (W, Ξ) is dlt (resp. B-smooth) by the adjunction.

If (X, Δ) is dlt pair or B-smooth and W is a center of log canonical singularities, then we write $W \Subset X$.

Lemma 5.7. *Let (X, Δ) be a pure n -dimensional (not necessarily connected) proper dlt pair with $K_X + \Delta$ nef and let m be a sufficiently large and divisible integer. We write $G = \rho_m(\text{Bir}(X, \Delta))$. If $s \in \text{PA}(X, m(K_X + \Delta))$, then $g^*s \in \text{PA}(X, m(K_X + \Delta))$ for every $g \in G$ and $g^*s|_{\lfloor \Delta \rfloor} = s|_{\lfloor \Delta \rfloor}$.*

In particular if $|G|$ is finite, then

$$\sum_{g \in G} g^*s \in \text{A}(X, m(K_X + \Delta)),$$

$$\prod_{g \in G} g^*s \in \text{A}(S, m|G|(K_X + \Delta)),$$

and

$$\prod_{g \in G} g^*s|_{\lfloor \Delta \rfloor} = (s|_{\lfloor \Delta \rfloor})^{|G|}.$$

For the proof of this lemma we use the next two claims and Szabó's resolution lemma, which is a powerful tool of treating dlt pairs (see [4]).

Claim (A_n). Let (T, Θ) and (S, Ξ) be n -dimensional B-smooth pairs and $h : S \rightarrow T$ a B-birational morphism. If $W \Subset T$, then there is a $W' \Subset S$ such that $W' \rightarrow W$ is a B-birational morphism.

Claim (B_n). Let (T, Θ) and (S, Ξ) be n -dimensional B-smooth pairs and $h : S \rightarrow T$ a B-birational morphism. Assume that $W \Subset S$. If $W \rightarrow h(W)$ is not B-birational, then there is a $W' \Subset W$ such that $W' \rightarrow h(W)$ is a B-birational morphism and that the inclusion $W' \rightarrow W$ induces the isomorphism $H^0(W, \mathcal{O}_W(m(K_S + \Xi)|_W)) \simeq H^0(W', \mathcal{O}_{W'}(m(K_S + \Xi)|_{W'}))$.

By Proposition 5.3, we have the following results.

Corollary 5.8 (cf. [14, 12.1.1, 8.5]). *Let (X, Δ) be a proper slc n -fold such that $K_X + \Delta$ is nef, where $n \leq 2$. Then $|m(K_X + \Delta)|$ is free for a sufficiently large and divisible integer m . In particular, if (X, Δ) is proper sdlt, then $\mathcal{O}_X(m(K_X + \Delta))$ is generated by $\text{A}(X, m(K_X + \Delta))$.*

The next corollary is the main theorem of this paper (see Theorem 1.1).

Corollary 5.9. *Let (X, Δ) be a proper slc threefold with $K_X + \Delta$ nef. Then $K_X + \Delta$ is semiample.*

We can reformulate Corollaries 5.8 and 5.9 in the following form if we list all the necessary results (e.g. log MMP) as the assumption.

Theorem 5.10. *Assume the log MMP for dimension n . If the abundance conjecture holds for lc n -folds and if the finiteness of B-pluricanonical representations (see Conjecture 4.2) is true for dimension n , then the abundance conjecture is true for sdlt and slc n -folds, that is, Corollary 5.8 holds in dimension n .*

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