Homotopy groups of compact exceptional Lie groups

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We consider homotopy groups of compact, connected, simply connected simple exceptional Lie groups G; they are classified as G_2, F_4, E_6, E_7 and E_8. By the Hopf theorem we have

$$H^*(G;Q) \cong \Lambda(x_1, x_2, \cdots, x_\ell),$$

where ℓ is the rank of G, $\deg x_i$ is odd and $\sum_{i=1}^{\ell} \deg x_i$ is the dimension of G. Let X be the direct product of spheres of dimensions $\deg x_1, \cdots, \deg x_\ell$, then, from Serre's $\mathcal{C}$-theory [10], $\pi_i(G)$ are $\mathcal{C}$-isomorphic to $\pi_i(X)$ for all i, where $\mathcal{C}$ is the class of finite abelian groups. Therefore the rank of $\pi_q(G)$ is equal to the number of such i that $\deg x_i$ is equal to q, particularly if q is even, then $\pi_q(G)$ is finite. It is a classical fact that $\pi_2(G) = 0$ and $\pi_3(G) \cong Z$.

According to Bott-Samelson [1], we have

$$\begin{aligned}
&\pi_i(E_6) = 0 \text{ for } 4 \le i \le 8, && \pi_9(E_6) \cong Z,\\
&\pi_i(E_7) = 0 \text{ for } 4 \le i \le 10, && \pi_{11}(E_7) \cong Z,\\
&\pi_i(E_8) = 0 \text{ for } 4 \le i \le 14, && \pi_{15}(E_8) \cong Z.
\end{aligned}$$

For $i \le 23$, the homotopy groups $\pi_i(G_2)$ and $\pi_i(F_4)$ are determined in [7] by the second author. For $G = E_6$, E_7 and E_8, the 2-primary components of homotopy group $\pi_i(G)$ are determined in [5] by the first author for $i \le 22$, $i \le 25$ and $i \le 28$ respectively.

The purpose of this paper is to calculate $\pi_i(G : 3)$ the 3-primary components of homotopy groups $\pi_i(G)$ by making use of usual methods, such as the killing-homotopy method [2] and homotopy exact sequences associated with various fibrations.

Let $B_1(2n-1, 2n+3)$ be a S^{2n-1}-bundle over S^{2n+3} with the characteristic element $\alpha_1(2n-1) \in \pi_{2n+2}(S^{2n-1} : 3)$ so that $H^*(B_1(2n-1, 2n+3); Z_3) \cong \Lambda(x_{2n-1}, x_{2n+3})$, where $x_{2n+3} = \wp^1 x_{2n-1}$. Let $B_2(2n-1, 2n+7)$ be a S^{2n-1}-bundle over S^{2n+7} with the characteristic element $\alpha_2(2n-1) \in \pi_{2n+6}(S^{2n-1} : 3)$ so that $H^*(B_2(2n-1, 2n+7); Z_3) \cong \Lambda(x_{2n-1}, x_{2n+7})$, where $x_{2n+7} = \Phi x_{2n-1}$; Φ is the secondary cohomology operation associated with the relation $\wp^1\beta\wp^1 - \beta\wp^2 - \wp^2\beta = 0$.

Homotopy group of G_2. It is known [7] that G_2 is 3-equivalent to $B_2(3, 11)$, and so $\pi_i(G_2)$ are $\mathcal{C}_3$-isomorphic to $\pi_i(B_2(3, 11))$, where $\mathcal{C}_3$ is the class of finite abelian groups without 3 torsion. Thus $\pi_i(G_2 : 3)$ can be calculated from the homotopy exact sequence

$$\begin{aligned}
\cdots \to \pi_{i+1}(S^{11}) \to \pi_i(S^3) \to \pi_i(B_2(3, 11))\\
\to \pi_i(S^{11}) \to \pi_{i-1}(S^3) \to \cdots
\end{aligned}$$

associated with the bundle $B_2(3, 11)/S^3 = S^{11}$.

Homotopy group of F_4. Let K be a finite complex constructed by Harper in [3] such that

$$H^*(K; Z_3) \cong Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7),$$

where $x_7 = \wp^1 x_3$ and $x_8 = \beta x_7$. Then, according to Harper [3], F_4 is 3-equivalent to $K \times B_1(11, 15)$.

We consider the 3-connective fiber space $\tilde{F}_4$ over F_4 by killing $\pi_3(F_4)$ so that $\pi_i(\tilde{F}_4) \cong \pi_i(F_4)$ for $i \ge 4$. Then we have that $\tilde{F}_4$ is 3-equivalent to $\tilde{K} \times B_1(11, 15)$, where $\tilde{K}$ is the 3-connective fiber space over K and $H^*(\tilde{K}; Z_3) \cong Z_3[y_{18}] \otimes \Lambda(y_{19}, y_{23})$, where $y_{19} = \beta y_{18}$ and $y_{23} = \wp^1\beta y_{18}$. Now we approximate $\tilde{K}$ by a finite cell complex $K' = S^{18} \cup_{3\iota_{18}} e^{19} \cup e^{23}$ such that $\pi_i(\tilde{K})$ is $\mathcal{C}_3$-isomorphic to $\pi_i(K')$ for $i \le 34$.

Then we have the homotopy exact sequence

$$\begin{aligned}
\cdots \to \pi_i(S^{22}) \xrightarrow{f_*} \pi_i(S^{18} \cup_{3\iota_{18}} e^{19}) \to \pi_i(K')\\
\to \pi_{i-1}(S^{22}) \xrightarrow{f_*} \pi_{i-1}(S^{18} \cup_{3\iota_{18}} e^{19}) \to \cdots
\end{aligned}$$

for $i \le 34$, where $f : S^{22} \to S^{18} \cup_{3\iota_{18}} e^{19}$ is the attaching map for e^{23}. Thus we can obtain $\pi_i(F_4 : 3)$ for $i \le 34$. (Recall that $\pi_i(B_1(11, 15)$ is calculated in [9]).

Homotopy group of E_6. According to Harris [4], E_6 is 3 equivalent to $F_4 \times B_2(9, 17)$. Thus $\pi_i(E_6 : 3)$ can be read off from those of F_4 and $B_2(9, 17)$, which is calculated from the homotopy exact sequence associated with the bundle $B_2(9, 17)/S^9 = S^{17}$.

Homotopy group of E_8. The 3-connective fiber space $\tilde{E}_8$ over E_8 has the following mod 3 co-

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homology [6]:

$$H^*(\tilde{E_8}; Z_3)$$
$$\cong Z_3[x_{54}] \otimes \Lambda(x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}, x_{55}, x_{59}),$$

where $\Phi x_{15} = x_{23}, \wp^1 x_{23} = x_{27}$ and $\Phi x_{27} = \pm x_{35}$. Then there exists a cell complex $K_8 = S^{15} \cup_{\alpha_2} e^{23} \cup e^{27} \cup e^{35}$ such that $\pi_i(K_8)$ is $\mathcal{C}_3$-isomorphic to $\pi_i(\tilde{E_8})$ for $i \leq 36$, where $\alpha_2 = \alpha_2(15)$. Let $f : S^{26} \to K_{\alpha_2} = S^{15} \cup_{\alpha_2} e^{23}$ and $g : S^{34} \to K_{\alpha_2} \cup e^{27}$ be the attaching maps for e^{27} and e^{35} respectively. Then $p \circ f = \alpha_1(23)$ and $p' \circ g = \alpha_2(27)$, where $p : K_{\alpha_2} \to S^{23}$ and $p' : K_{\alpha_2} \cup e^{27} \to S^{27}$ are the maps which shrink S^{15} and K_{α_2} respectively to a point. The 3-primary component of $\pi_i(K_8)$ for $i \leq 36$ are calculated by means of the following three exact sequences;

$$\cdots \to \pi_i(S^{22}) \overset{\alpha_{2*}}{\to} \pi_i(S^{15}) \to \pi_i(K_{\alpha_2})$$
$$\to \pi_{i-1}(S^{22}) \overset{\alpha_{2*}}{\to} \pi_{i-1}(S^{15}) \to \cdots,$$
$$\cdots \to \pi_i(S^{26}) \overset{f_*}{\to} \pi_i(K_{\alpha_2}) \to \pi_i(K_{\alpha_2} \cup e^{27})$$
$$\to \pi_{i-1}(S^{26}) \overset{f_*}{\to} \pi_{i-1}(K_{\alpha_2}) \to \cdots,$$
$$\cdots \to \pi_i(S^{34}) \overset{g_*}{\to} \pi_i(K_{\alpha_2} \cup e^{27}) \to \pi_i(K_8)$$
$$\to \pi_{i-1}(S^{34}) \overset{g_*}{\to} \pi_{i-1}(K_{\alpha_2} \cup e^{27})$$
$$\to \cdots.$$

Homotopy group of E_7. We obtain that $H^*(E_7/F_4; Z_3) \cong \Lambda(z_{19}, z_{27}, z_{35})$ where $\Phi z_{19} = z_{27}$ and $\Phi z_{27} = z_{35}$. Then there exists a cell complex $K_{E_7/F_4} = S^{19} \cup_{\alpha_2} e^{27} \cup e^{35}$ such that $\pi_i(K_{E_7/F_4})$ are $\mathcal{C}_3$-isomorphic to $\pi_i(E_7/F_4)$ for $i \leq 44$. This enables us to calculate the 3-primary components of $\pi_i(E_7/F_4)$ for $i \leq 44$.

On the other hand we obtain that $H^*(\Omega(E_8/E_7); Z_3) \cong \Lambda(w_{11}, w_{19})$ for deg ≤ 37. Then there exists a cell complex $K_{7,8} = S^{11} \cup e^{19} \cup e^{30}$ such that $\pi_i(K_{7,8})$ are $\mathcal{C}_3$-isomorphic to $\pi_i(E_8/E_7)$ for $i \leq 36$.

By making use of the homotopy exact sequences associated with the fibrations $F_4 \to E_7 \to E_7/F_4$ and $\Omega(E_8/E_7) \to E_7 \to E_8$, we can calculate the 3-primary component of $\pi_i(E_7)$.

Theorem 1. *The* 3*-primary component of* $\pi_i(G)$ *are summarized in the following Table.*

Table

i	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\pi_i(G_2:3)$	∞	0	0	3	0	0	3	0	∞	0	0	3	0	3
$\pi_i(F_4:3)$	∞	0	0	0	0	0	0	0	∞	0	0	0	∞	0
$\pi_i(E_6:3)$	∞	0	0	0	0	0	∞	0	0	3	0	0	0	0
$\pi_i(E_7:3)$	∞	0	0	0	0	0	0	0	∞	0	0	0	∞	0
$\pi_i(E_8:3)$	∞	0	0	0	0	0	0	0	0	0	0	0	∞	0

i	17	18	19	20	21	22	23	24	25
$\pi_i(G_2:3)$	0	3	3	0	0	9	0	3	0
$\pi_i(F_4:3)$	0	9+3	0	0	$(3)^2$	27	∞	0	0
$\pi_i(E_6:3)$	∞	9+3	3	27	3+3	27+3	∞	9	0
$\pi_i(E_7:3)$	0	3	∞	0	3	27	∞	0	0
$\pi_i(E_8:3)$	0	3	0	0	0	0	∞	0	3

i	26	27	28	29	30
$\pi_i(G_2:3)$	$(3)^2$	3	0	3	3
$\pi_i(F_4:3)$	27+3	0	$(3)^2$	3	3+9
$\pi_i(E_6:3)$	27+3	3+3	$27+(3)^2$	$(3)^2$	$9+(3)^2$
$\pi_i(E_7:3)$	9 or 3	∞	3	0	27+9
$\pi_i(E_8:3)$	0	∞	3	0	27

i	31	32	33	34	35	36
$\pi_i(G_2:3)$	0	0	3	9+3	3	3
$\pi_i(F_4:3)$	$(3)^2$	3	$(3)^2$	27+3		
$\pi_i(E_6:3)$	$9+(3)^2$	$27+(3)^2$	3	27+3		
$\pi_i(E_7:3)$	3	3	$(3)^2$			
$\pi_i(E_8:3)$	0	0	3	0	∞+3	0

In the above table an integer n indicates a cyclic group Z_n of order n, the symbol ∞ an infinite cyclic group Z, the symbol + the direct sum of the group, and $(3)^k$ indicates the direct sum of k-copies of Z_3. Some of our results were announced in [13].

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