K-approximations and strongly countable-dimensional spaces

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1. Introduction. Throughout the present paper, by the dimension we mean the covering dimension dim. We shall consider a characterization of a class of infinite dimensional metrizable spaces in terms of K-approximations. In [5], Dydak-Mishra-Shukla introduced a concept of a K-approximation of a mapping to a metric simplicial complex and characterized n-dimensional spaces and finitistic spaces in terms of K-approximations. Let X be a space, K a metric simplicial complex and $f : X \to K$ a continuous mapping. A mapping $g : X \to K$ is said to be a *K-approximation of f* if for each simplex $\sigma \in K$ and each $x \in X$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. A K-approximation $g : X \to K$ of f is called an *n-dimensional K-approximation* if $g(X) \subset K^{(n)}$ and a *finite dimensional K-approximation* if $g(X) \subset K^{(m)}$ for some natural number m, where $K^{(m)}$ denotes the m-skelton of K.

The concept of finitistic spaces was introduced by Swan [12] for working in fixed point theory and is applied to the theory of transformation groups by using the cohomological structures (cf. [1]). For a family $\mathcal{U}$ of a space X the order $\operatorname{ord}\mathcal{U}$ of $\mathcal{U}$ is defined as follows: $\operatorname{ord}_x\mathcal{U} = |\{U \in \mathcal{U} : x \in U\}|$ for $x \in X$ and $\operatorname{ord}\mathcal{U} = \sup\{\operatorname{ord}_x\mathcal{U} : x \in X\}$. We say a family $\mathcal{U}$ has *finite order* if $\operatorname{ord}\mathcal{U} = n$ for some natural number n. A space X is said to be *finitistic* if every open cover of X has an open refinement with finite order. We notice that finitistic spaces are also called *boundedly metacompact spaces* (cf. [7]). It is obvious that all compact spaces and all finite dimensional paracompact spaces are finitistic spaces. More precisely, we have a useful characterization of finitistic spaces.

Proposition ([5], [8]). *A paracompact space X is finitistic if and only if there is a compact subspace C of X such that $\dim F < \infty$ for every closed subspace F with $F \cap C = \emptyset$.*

The dimension-theoretic properties of finitistic spaces are investigated by several authors (cf. [3], [4], [5] and [8]). In particular, Dydak-Mishra-Shukla ([5]) proved the following.

Theorem A ([5]). *For a paracompact space X the following are equivalent.*

(a) $\dim X \le n$.
(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f.*
(c) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f such that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.*

Theorem B ([5]). *For a paracompact space X the following are equivalent.*

(a) *X is a finitistic space.*
(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f.*
(c) *For every integer $m \ge -1$, every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f such that $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

The purpose of the present note is to extend Theorem A to a class of metrizable spaces that have strong large transfinite dimension.

For a metric space (X, ρ), a subset A of X and $\varepsilon > 0$ we denote $S_\varepsilon(A) = \{x \in X : \rho(x, A) < \varepsilon\}$. We denote the set of natural numbers by ω. We refer the reader to [6] and [11] for basic results in dimension theory.

2. Results. We begin with the definition of strong small transfinite dimension introduced by Borst [2]. A normal space X is said to *have strong small transfinite dimension* if for every non-empty

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closed set F of X there is an open normal subspace U of F such that $\dim \overline{U} < \infty$. (We notice that spaces that have strong small transfinite dimension are called *shallow spaces* in [6].) Recall from [10] that a normal space X *has strong large transfinite dimension* if X has both large transfinite dimension and strong small transfinite dimension. We use the following characterization of spaces that have strong large transfinite dimension. A normal space X is said to be *strongly countable-dimensional* if X is a union of countably many finite dimensional closed subsets.

Lemma 1 ([9, Proposition 2.2 and 2.3]). *Let X be a metrizable space. Then X has strong large transfinite dimension if and only if X is finitistic and strongly countable-dimensional.*

The following is a main result of the paper. For a space X we denote $\mathcal{D}(X) = \{D : D$ is a closed discrete subset of $X\}$.

Theorem. *For a metrizable space X the following are equivalent.*

(a) *X has strong large transfinite dimension.*
(b) *There is a function $\varphi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.*
(c) *For every integer $m \geq -1$ there is a function $\psi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Proof. (a) $\Rightarrow$ (b): By Lemma 1 and Proposition, there is a compact subset C of X such that $\dim F < \infty$ for each closed set F of X with $F \cap C = \emptyset$. For each $i \in \omega$ we put $H_i = X \setminus S_{1/i}(C)$ and $\dim H_i = m_i < \infty$. Since C is strongly countable-dimensional, there is a countable closed cover $\{C_i : i \in \omega\}$ of C such that $C_i \subset C_{i+1}$ and $\dim C_i = n_i < \infty$ for each i. Let $D \in \mathcal{D}(X)$. Since C is compact, there is i such that $C \cap D \subset C_i$. On the other hand, there is j such that $j \geq i$ and $D \setminus C \subset H_j$. Then we put $\varphi(D) = \sum_{k=1}^{j}(n_k + m_k) + 2j$. Let K be a metric simplicial complex and $f : X \to K$ a continuous mapping. For each vertex v of K let $\mathrm{St}(v, K)$ be the union of geometric interiors of all simplexes of K containing v as a vertex. Then $\{\mathrm{St}(v, K) : v \in K^{(0)}\}$ is an open cover of K. It follows from an argument similar to [9, Theorem 3.6] that there are locally finite families of open sets $\mathcal{U}_k$ and $\mathcal{V}_k$, $k \in \omega$, of X ($\mathcal{U}_k$ and $\mathcal{V}_k$ need not cover X) which satisfy the following conditions:

(1) $C_k \setminus \bigcup\{C_l : l < k\} \subset \bigcup \mathcal{U}_k \subset X \setminus (H_k \cup (\bigcup\{C_l : l < k\}))$.
(2) $H_k \setminus \bigcup\{H_l : l < k\} \subset \bigcup \mathcal{V}_k \subset X \setminus (\overline{S_{1/k}(C)} \cup (\bigcup\{H_l : l < k\}))$.
(3) $\operatorname{ord} \mathcal{U}_k \leq n_1 + n_2 + \cdots + n_k + k$.
(4) $\operatorname{ord} \mathcal{V}_k \leq m_1 + m_2 + \cdots + m_k + k$.
(5) $\mathcal{U}_k$ and $\mathcal{V}_k$ are refinements of $\{f^{-1}(\mathrm{St}(v, K)) : v \in K^{(0)}\}$.

Then $\mathcal{W} = \bigcup_{k=1}^{\infty} \mathcal{U}_k \cup \bigcup_{k=1}^{\infty} \mathcal{V}_k$ is an open cover of X such that $\sup\{\operatorname{ord}_x \mathcal{W} : x \in D\} \leq \varphi(D)$ for each $D \in \mathcal{D}(X)$. For each $W \in \mathcal{W}$ there is $v(W) \in K^{(0)}$ such that $W \subset f^{-1}(\mathrm{St}(v(W), K))$. Let $\mathcal{P}$ be a locally finite open refinement of $\mathcal{W}$. For each $P \in \mathcal{P}$ there is $W(P) \in \mathcal{W}$ such that $P \subset W(P)$. Put $v(P) = v(W(P))$ for each $P \in \mathcal{P}$. For each $v \in K^{(0)}$ we put $Q_v = \bigcup\{P \in \mathcal{P} : v(P) = v\}$, and $\mathcal{Q} = \{Q_v : v \in K^{(0)}\}$. Then $\mathcal{Q}$ is a locally finite open cover of X such that $Q_v \subset f^{-1}(\mathrm{St}(v, K))$ for each $v \in K^{(0)}$ and $\sup\{\operatorname{ord}_x \mathcal{Q} : x \in D\} \leq \varphi(D)$ for each $D \in \mathcal{D}(X)$. Let $\{\kappa_v : v \in K^{(0)}\}$ be a partition of unity subordinated to $\mathcal{Q}$. We define $g : X \to K$ as $g(x) = \sum_{v \in K^{(0)}} \kappa_v(x) \cdot v$, $x \in X$. It is easy to see that g is a K-approximation of f and $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.

(b) $\Rightarrow$ (a): For each $x \in X$ let $\varphi(x) = \varphi(\{x\})$. To show that X is strongly countable-dimensional, let $\mathcal{U}$ be an open cover of X. By an argument similar to [5, Theorem 2.1], we have an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\operatorname{ord}_x \mathcal{V} \leq \varphi(x) + 1$ for each $x \in X$. For each n we put $A_n = \{x \in X : \varphi(x) \leq n\}$ and $X_n = \overline{A_n}$. It follows that $X = \bigcup_{n=1}^{\infty} X_n$ and each X_n is closed subset of X with $\dim X_n \leq n$ (cf. [6, Theorem 5.1.10]). Next, we suppose that X is not finitistic. Then there is an open cover $\mathcal{U}$ of X such that for every open refinement $\mathcal{V}$ of $\mathcal{U}$ $\sup\{\operatorname{ord}_{x_n} \mathcal{V} : n \in \omega\} = \infty$ for some sequence $A = \{x_n : n \in \omega\}$ in X. By an argument similar to [5, Theorem 2.1], it follows that there is a locally finite open refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\sup\{\operatorname{ord}_x \mathcal{W} : x \in D\} \leq \varphi(D)$ for each $D \in \mathcal{D}(X)$. Hence A is not closed discrete in X and hence A has an accumulation point x_0. Then $\operatorname{ord}_{x_0} \mathcal{W} = \infty$. This contradicts the local finiteness of $\mathcal{W}$. Therefore, X is a finitistic space and hence, by Lemma 1, X has strong large transfinite dimension.

To show the implication (a) $\Rightarrow$ (c), we need the following.

Lemma 2 ([5, Corollary 1.7]). *Let $f : X \to K$ be a continuous mapping of a normal space X to a metric simplicial complex K, A is a subset of X, n a non-negative integer such that $f(A) \subset K^{(n)}$. Then, there are an open set U of X and a K-approximation g of f such that $A \subset U$, $g|A = f|A$ and $g|U$ is an n-dimensional K-approximation of $f|U$.*

(a) ⇒ (c): Let $\varphi : \mathcal{D}(X) \to \omega$ be as in (b). We put $\psi(D) = \max\{m, \varphi(D)\}$ for each $D \in \mathcal{D}(X)$. Let K be a metric simplicial complex and $f : X \to K$ a continuous mapping. By Lemma 2, there are an open set U of X and a K-approximation g_1 of f such that $f^{-1}(K^{(m)}) \subset U$, $g_1|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$ and $g_1|U$ is an m-dimensional K-approximation of $f|U$. Then, by (b), there is a K-approximation g_2 of g_1 such that $g_2(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$. Since X is finitistic, it follows from Theorem B that there is a finite dimensional K-approximation g_3 of g_2. Then $g_3(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$. Let $\kappa : X \to [0,1]$ be a continuous mapping such that $\kappa(f^{-1}(K^{(m)})) = 1$ and $\kappa(X \setminus U) = 0$. We define $g(x) = \kappa(x) \cdot g_1(x) + (1 - \kappa(x)) \cdot g_3(x)$ for each $x \in X$. It is easy to see that g is desired.

(c) ⇒ (b) is obvious. This completes the proof. □

By the proof of the theorem, we have the following.

Corollary. *For a paracompact space X the following are equivalent.*

(a) *X is a strongly countable-dimensional space.*

(b) *There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.*

(c) *For every integer $m \geq -1$ there is a function $\psi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

We do not know whether the theorem holds for paracompact spaces.

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