A generalization of the Hardy theorem to semisimple Lie groups

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1. Introduction. The classical uncertainty principle asserts that a function and its Fourier transform cannot both be concentrated on intervals of small measure. In the case of the Euclidean space, various forms of the uncertainty principle are known. One of them is the following Hardy theorem (cf. [1, pp. 155-158]). If a measurable function f on $\mathbf{R}$ satisfies $|f| \leq C\exp\{-ax^2\}$ and $|\hat{f}| \leq C\exp\{-by^2\}$ and $ab > 1/4$, then $f = 0$ (a.e.). Here we take $\hat{f}(y) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)\exp\{\sqrt{-1}xy\}dx$ as the definition of the Fourier transform of f.

Recently A. Sitaram and M. Sundari [6] generalized this theorem to the cases of the semisimple Lie groups with one conjugacy class of Cartan subgroups, the Riemannian symmetric spaces and $SL(2,\mathbf{R})$. On the other hand M. Sundari [7] showed the Hardy theorem for the Euclidean motion group. And also M. Eguchi, S. Koizumi and K. Kumahara generalized the results to Cartan motion group [2] and gave an L^p version for motion groups [3]. In this paper we give an analogue of the Hardy theorem to noncompact semisimple Lie groups.

2. Notation and preliminaries. The standard symbols $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ shall be used for the set of the integers, the real numbers and the complex numbers. If V is a vector space over $\mathbf{R}$, V_c, V^* and V_c^* denote its complexification, its real dual and its complex dual, respectively. For a Lie group L, $\hat{L}$ denotes the set of all equivalence classes of irreducible unitary representations of L. If L is a reductive Lie group, $\hat{L}_{\mathrm{disc}}$ denotes the subset comprised of all equivalence classes of discrete series. As usual for a Lie group, we use lower case German letters to denote its Lie algebras. If $\mathcal{H}$ is a complex separable Hilbert space, the operator norm on $\mathcal{H}$ will be denoted by $\|\cdot\|_\infty$.

Let G be a connected semisimple Lie group with finite center. We fix a maximal compact subgroup K of G and denote by θ the corresponding Cartan involution. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ defined by θ. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and denote by Σ the set of all restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$. We fix an order in $\mathfrak{a}_\mathfrak{p}^*$ and denote by Σ^+ the set of all positive restricted roots. Let $\{\alpha_1, \cdots, \alpha_l\}$ be the set of simple roots of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ and put $L = \left\{ \sum_{i=1}^l n_i\alpha_i \ ; \ n_i \in \mathbf{Z} \quad (i = 1, \cdots, l)\right\}$ and $L^+ = \left\{\sum_{i=1}^l n_i\alpha_i \ ; \ n_i \in \mathbf{Z}_{\geq 0} \quad (i = 1, \cdots, l)\right\}$. For each $\lambda = \sum_{i=1}^l \lambda_i\alpha_i \in L^+$, put $|\lambda| = \sum_{i=1}^l \lambda_i$. For $q = (q_1, \ldots, q_l) \in \mathbf{Z}_{\geq 0}^l$, we also put $\alpha(H)^q = \alpha_1(H)^{q_1}\cdots\alpha_l(H)^{q_l} \quad (H \in \mathfrak{a})$ and $|q| = \sum_{i=1}^l q_i$. Let $\mathfrak{a}_\mathfrak{p}^-$ be a choice of negative Weyl chamber. We set $\mathfrak{n}_\mathfrak{p} = \sum_{\lambda\in\Sigma^+}\mathfrak{g}_\lambda$ and put $A_\mathfrak{p} = \exp\mathfrak{a}_\mathfrak{p}$, $N_\mathfrak{p} = \exp\mathfrak{n}_\mathfrak{p}$ and $M_\mathfrak{p} = Z_K(\mathfrak{a}_\mathfrak{p})$. Then $P_0 = M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ is a minimal parabolic subgroup of G. When $g = k\exp X \quad (k \in K, X \in \mathfrak{p})$, we set $\sigma(g) = \|X\|$, where $\|\cdot\|$ denoting the norm coming from the Killing form of $\mathfrak{g}$.

We write $\mathrm{Car}(G)$ for the set of all θ-stable Cartan subgroups and denote by $\mathrm{Car}'(G)$ the subset of $\mathrm{Car}(G)$ comprised of all noncompact ones. For $J \in \mathrm{Car}(G)$, let $P_J = M_JA_JN_J$ be the Langlands decomposition of the cuspidal parabolic subgroup P_J associated to J. We remark that if $J \in \mathrm{Car}'(G)$ then $A_J \neq \{e\}$.

Under the decomposition $G{=}KP_J{=}KM_JA_JN_J$, each $g{\in}G$ can be written as $g{=}\kappa(g)\mu_J(g)\exp H_J(g)\, n_J(g)$, where $\kappa(g) \in K$, $\mu_J(g) \in M_J$, $H_J(g) \in \mathfrak{a}_J$ and $n_J(g) \in N_J$. Let $\sigma \in (\hat{M}_J)_{\mathrm{disc}}$ and $\nu \in \mathfrak{a}_J^*$. We denote by $(\pi_{J,\sigma,\nu}, \mathcal{H}^{J,\sigma,\nu})$ the representation induced from $\sigma\otimes\nu\otimes 1$ of P_J to G. Then it is known that $(\pi_{J,\sigma,\nu}, \mathcal{H}^{J,\sigma,\nu})$ is unitary.

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3. The main theorem. As the Hardy theorem in the case $\hat{G}_{\rm disc} = \emptyset$ is obtained in [6], we suppose rank$G$ = rank$K$, that is, $\hat{G}_{\rm disc} \neq \emptyset$. Let $J \in {\rm Car}'(G)$ and $P_J = M_J A_J N_J$ be the corresponding cuspidal parabolic subgroup of G. For $\sigma \in (\hat{M}_J)_{\rm disc}$ and $\nu \in \mathfrak{a}_J^*$, we define the Fourier transform $\hat{f}_J(\sigma, \nu)$ of $f \in L^1(G)$ by

$$\hat{f}_J(\sigma, \nu) = \int_G f(g) \pi_{J,\sigma,\nu}(g)\, dg.$$

We write π_λ for the discrete series representation of G with Harish-Chandra parameter λ. We define the Fourier transform $\hat{f}_d(\lambda)$ of $f \in L^1(G)$ by

$$\hat{f}_d(\lambda) = \int_G f(g) \pi_\lambda(g)\, dg.$$

The following is an analogue of Hardy theorem for semisimple Lie group.

Theorem 3.1. *Let f be a measurable function on G such that*

$$(1) \qquad |f(g)| \le C e^{-a\sigma(g)^2}$$

$$(2) \qquad \|\hat{f}_J(\sigma, \nu)\|_\infty \le C_{J,\sigma} e^{-b\|\nu\|^2} \quad (\nu \in \mathfrak{a}_J^*)$$

for each $J \in {\rm Car}'(G)$, some constants $a > 0$, $b > 0$, $C > 0$ and $C_{J,\sigma} > 0$. If $ab > 1/4$ then $f = 0$ (a.e.).

We give here a sketch of the proof. From the assumption for the function f, we see that $f \in L^1(G) \cap L^2(G)$ and $\hat{f}_J(\sigma, \nu)$ makes sense on $(\hat{M}_J)_{\rm disc} \times (\mathfrak{a}_J^*)_c$. It is shown that the assumption of f and $\hat{f}_J$ in Theorem 3.1 implies $\hat{f}_J(\sigma, \nu) = 0$ for $\nu \in \mathfrak{a}_J^*$. This is accomplished by a similar argument to that of [6]. By the Plancherel formula and the above argument for continuous series, we can suppose $f \in L_d^2(G)$, $L_d^2(G)$ denoting the closed subspace of $L^2(G)$ spanned by the set of K-finite matrix elements of discrete series of G. Decomposing f into the sum

$$f = \sum \chi_{\tau_1} * f * \chi_{\tau_2} = \sum f_{\tau_1, \tau_2}$$

and using the assumptions (1) and (2), we can prove that f_{τ_1,τ_2} satisfies also the assumptions (1) and (2). Therefore, we can assume

$$f = \sum_{1 \le k \le s} C_k \Phi^k,$$

where Φ^k is a matrix element of a discrete series. We apply the asymptotic expansion to each Φ^k and pay attention that the leading exponents are of first order. Then we can see the assumption (2) on f leads to a contradiction by the following theorem.

Theorem 3.2 ([4,5]). *Let Φ be a matrix element of a discrete series. Then there exist mutually integrally non-equivalent elements $\mu_1, \cdots, \mu_r \in \mathfrak{a}_J^*$, a finite set $\mathcal{M} \subset \mathbf{Z}_{\ge 0}^l$ and constants $C_{\mu_j+\lambda,q} \in \mathbf{C}$ $(j = 1, \cdots, r,\ \lambda \in L^+,\ q \in \mathcal{M})$ with $C_{\mu_j,q} \neq 0$ $(j = 1, \cdots, r,\ q \in \mathcal{M})$, such that*

$$\Phi(a) = \sum_{j=1}^r \sum_{q \in \mathcal{M}} \alpha(\log a)^q e^{(\mu_j+\rho)(\log a)} \times \sum_{\lambda \in L^+} C_{\mu_j+\lambda,q} e^{\lambda(\log a)}, \quad a \in A_{\mathfrak{p}}^-.$$

Here the series on the right hand side is absolutely and uniformly convergent as long as $a \in A_{\mathfrak{p}}^-$, $-\alpha_j(\log a) \ge \varepsilon_j > 0$.

References

[1] H. Dym and H. P. McKean: Fourier series and Integral. Academic Press, New York, pp. 155–158 (1972).

[2] M. Eguchi, S. Koizumi and K. Kumahara: An analogue of the Hardy theorem for the Cartan motion group. Proc. Japan Acad., **74A**, 149–151 (1998).

[3] M. Eguchi, S. Koizumi and K. Kumahara: An L^p version of the Hardy theorem for motion groups. J. Austral. Math. Soc. (in print).

[4] Harish-Chandra: Some results on differential equations (1960) (manuscript).

[5] D. Miličić: Asymptotic behaviour of matrix coefficients of the discrete series. Duke Math. J., **44**, 59–88 (1977).

[6] A. Sitaram and M. Sundari: An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups. Pacific J. Math., **177**, 187–200 (1997).

[7] M. Sundari: Hardy's theorem for the n-dimensional Euclidean motion group. Proc. Amer. Math. Soc., **126**, 1199–1204 (1998).