"Hasse principle" for free groups

By Takashi ONO *) and Hideo WADA **)

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1. Notation and results. Let G be a group and f be a cocyle, i.e., a mapping from G to G which satisfies

(1) $f(XY) = f(X)f(Y)^X$ *for any* $X, Y \in G$ *where* $Y^X = XYX^{-1}$.

For each $X \in G$, if there exists $M \in G$ such that $f(X) = M^{-1}M^X$, then f is called a *local coboundary.* More strongly, if M can be chosen independent of X, then f is called a *global coboundary.* If any local coboundary is a global coboundary, we say that G enjoys the *Hasse principle.*

For any integer $N \geq 1$, we set

$$\Gamma(N) = \{A \in SL_2(\mathbf{Z}); A \equiv E \bmod N\}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, we set

$$\bar{\Gamma}(N) = \begin{cases} \Gamma(N)/\{\pm E\} & \text{if } N = 1, 2, \\ \Gamma(N) & \text{if } N \geq 3. \end{cases}$$

In [5] it is proved that $G = PSL_2(\mathbf{Z}) = \bar{\Gamma}(1)$ and $G = PSL_2(\mathbf{F}_p)$ enjoy the Hasse principle. In this paper we shall prove the following

Theorem. *Any free group of finite rank enjoys the Hasse principle.*

For $N \geq 2$, $\bar{\Gamma}(N)$ is a free group of finite rank (cf. [1] p.362, 3D, Theorem). Therefore we get the following

Corollary. *For any* $N \geq 1$, $\bar{\Gamma}(N)$ *enjoys the Hasse principle.*

It is curious that we need parabolic matrices in $\Gamma(p)$, $p =$ an odd prime, to prove a theorem on free groups.

2. Proof of the theorem. Let p be an odd prime. Then $\Gamma(p)$ has $(p^2-1)/2$ parabolic elements and the following p parabolic elements

$$A = E + p\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$B_t = E + p\begin{pmatrix} t & -t^2 \\ 1 & -t \end{pmatrix}, t = 0, 1, \cdots, p-2,$$

which are independent because the cusps of A, B_t are ∞, t, respectively. Let $G = \langle A, B_0, B_1, \cdots, B_{k-2}\rangle$ be the free group generated by A, B_0, B_1, $\cdots$, B_{k-2}, $(2 \leq k \leq p)$, and f be a local coboundary. Then there is an element $M_1 \in G$ such that $f(A) = M_1^{-1}M_1^A$. Put $f_1(X) = M_1 f(X) M_1^{-X}$. Then f_1 is also a local coboundary and $f_1(A) = 1$. For any $B = B_t$ $(t \leq k-2)$, there exists $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ such that $f_1(B) = M^{-1}M^B = M^{-1}BMB^{-1}$. We can easily verify that

(2) $$M^{-1}BM = E + p\begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$$
where $x = a - ct$, $y = b - dt$.

As f_1 is a cocycle, we have

$$f_1(AB) = f_1(A)f_1(B)^A = A(M^{-1}BMB^{-1})A^{-1}.$$

On the other hand, since f_1 is a local coboundary, there exists $N_1 \in G$ such that

$$f_1(AB) = N_1^{-1}N_1^{AB} = N_1^{-1}ABN_1B^{-1}A^{-1}.$$

From these two equations, we get

(3) $$AM^{-1}BM = N_1^{-1}ABN_1$$

Taking the traces of matrices in (3), we have

$$\begin{aligned} &\mathrm{tr}(AM^{-1}BM) \\ &= \mathrm{tr}(E + p\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})(E + p\begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}) \\ &= \mathrm{tr}(E + p^2\begin{pmatrix} x^2 & xy \\ 0 & 0 \end{pmatrix}) = 2 + p^2x^2, \end{aligned}$$

$$\mathrm{tr}(N_1^{-1}ABN_1) = \mathrm{tr}(AB) = 2 + p^2.$$

Therefore x must be ± 1. As $x = a - ct \equiv 1 \pmod p$, we get $x = 1$.

If $t = 0$, then $a = x = 1$ and $y = b$. From (2), $M^{-1}BM$ depends only on x and y. So, if we put $M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then $M_2 \in G$ and

$$f_1(B) = M^{-1}BMB^{-1}$$

*) Department of Mathematics, Johns Hopkins University, U.S.A.

**) Department of Mathematics, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo 102–8554.

$$= M_2^{-1} B M_2 B^{-1} = M_2^{-1} M_2^B.$$

As $M_2 A = A M_2$, we get

$$f_1(A) = 1 = M_2^{-1} A M_2 A^{-1} = M_2^{-1} M_2^A.$$

Put $f_2(X) = M_2 f_1(X) M_2^{-X}$. Then f_2 is a local coboundary which satisfies

$$f_2(A) = f_2(B_0) = 1. \tag{4}$$

If $t > 0$, then we have

$$\begin{aligned} f_2(B_0 B) &= f_2(B_0) f_2(B)^{B_0} \\ &= B_0 (M^{-1} B M B^{-1}) B_0^{-1}, \\ &\quad \text{where } f_2(B) = M^{-1} M^B. \end{aligned}$$

(Note that the meaning of M is not the same as before). As f_2 is a local coboundary, there exists $N_2 \in G$ such that

$$f_2(B_0 B) = N_2^{-1} N_2^{B_0 B} = N_2^{-1} B_0 B N_2 B^{-1} B_0^{-1}.$$

Therefore we get

$$B_0 M^{-1} B M = N_2^{-1} B_0 B N_2. \tag{5}$$

Taking the traces of matrices in (5), we have, as in (2),

$$\begin{aligned} &\operatorname{tr}(B_0 M^{-1} B M) \\ &= \operatorname{tr}(E + p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})(E + p \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}) \\ &= \operatorname{tr}(E + p^2 \begin{pmatrix} 0 & 0 \\ -xy & -y^2 \end{pmatrix}) = 2 - p^2 y^2, \\ &\operatorname{tr}(N_2^{-1} B_0 B N_2) = \operatorname{tr}(B_0 B) = 2 - p^2(-t)^2. \end{aligned}$$

Therefore y must be $\mp t$. As $y = b - dt \equiv -t \pmod p$, we get $y = -t$. Since $x = 1$ and $y = -t$, we get

$$M^{-1} B M = E + p \begin{pmatrix} t & -t^2 \\ 1 & -t \end{pmatrix} = B, f_2(B) = 1.$$

So, from (4), we get $f_2(A) = f_2(B_0) = f_2(B_1) = \cdots = f_2(B_{k-2}) = 1$. Therefore $f_2(X) = 1$ for all $X \in G$. So we have

$$\begin{aligned} f(X) &= M_1^{-1}(M_2^{-1} f_2(X) M_2^X) M_1^X \\ &= (M_2 M_1)^{-1} (M_2 M_1)^X, \end{aligned}$$

and so f is a global coboundary.

For any free group G of rank k, we choose odd prime p such that $p \geq k$. Then G is isomorphic to $\langle A, B_0, B_1, \cdots, B_{k-2} \rangle$. Therefore G enjoys Hasse principle. Q.E.D.

References

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