On an identity of theta functions obtained from weight enumerators of linear codes*)

By Tomoaki MURAYAMA

Kakuda Senior High School, 1 Gyukan Kakuda, Kakuda, Miyagi 981-1505

(Communicated by Shokichi IYANAGA, M. J. A., June 23, 1998)

Abstract: D. P. Maher obtained an identity of theta functions by means of the Lee weight enumerators of linear codes over finite fields. S. S. Rangachari used it to prove an identity of theta functions conjectured by S. Ramanujan. In this paper, we consider the linear codes over finite rings, and give a generalization of Maher's identity. As an application, we generalize the result of Rangachari.

1. Introduction. Let a/n and b/n be rational numbers with $|a|, |b| < n$. We define the Jacobi theta function with a characteristic $(a/n, b/n)$ by

$$\vartheta_{a/n,b/n}(z, \tau) = \sum_{m\in Z}\exp\left\{\pi i\left(m+\frac{a}{n}\right)^2\tau + 2\pi i\left(m+\frac{a}{n}\right)\left(z+\frac{b}{n}\right)\right\}$$

for $z \in \boldsymbol{C}$, $\tau \in \boldsymbol{H} = \{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$.

In [3, p. 54] Ramanujan stated, without proof, a certain identity involving theta functions. Rangachari [4] has reformulated and proved this identity of Ramanujan in terms of the Jacobi theta functions:

Theorem(Ramanujan's identity). *For n odd,*

$$\sum_{r=-(n-1)/2}^{(n-1)/2}\vartheta_{r/n,0}(z, n\tau)^n = \vartheta_{0,0}(z, \tau)\cdot F(\tau).$$

For n even,

$$\sum_{r=-n/2+1}^{n/2}\vartheta_{r/n,0}(z, n\tau)^n = \vartheta_{0,0}(z, \tau)\cdot F(\tau)$$

with

$$F(\tau) = \sum_{m=0}^{\infty} a_m \exp(\pi i m\tau),$$

$$a_0 = 1,\ a_j = 0\ (0 < j < n-1),\ a_{n-1} = 2n.$$

Moreover Rangachari has proved the following identity of the theta function of the dual root lattice A_{n-1}^* (For the definition of this function, see below. cf. e.g. [5]. See also Remark 1.2 below):

Theorem(Rangachari). *If n is a prime number p, then*

$$F(\tau) = \Theta_{A_{p-1}^*}(p\tau).$$

To prove this theorem, Rangachari has essentially used the identity of Maher [2] between the Jacobi theta functions and theta functions of lattices defined by linear codes. The identity of Maher is obtained from the Lee weight enumerators of linear codes over a finite field $\boldsymbol{F}_p$.

Rangachari has also proved that this identity holds for the case of $n = 4$. He has conjectured that this identity would hold for any integer $n > 1$.

In this paper we extend the notion of linear codes from codes over a field $\boldsymbol{F}_p$ to codes over a ring $\boldsymbol{Z}/m\boldsymbol{Z}$ $(m > 1)$, and we define the Lee weight enumerators of linear codes over $\boldsymbol{Z}/m\boldsymbol{Z}$. As a result we extend the identity of Maher, and, by using this identity, we extend the result of Rangachari:

Theorem 1.1. *For any integer n greater than 1, we have*

$$F(\tau) = \Theta_{A_{n-1}^*}(n\tau).$$

Remark 1.2. In Rangachari's paper [4], the theta function $\Theta_{A_{p-1}^*}(\tau) = \sum_{x\in A_{p-1}^*} q^{x\cdot x}$, where $q = \exp(\pi i\tau)$, was defined by using Voronoi's principal form $x \cdot x = p\langle x, x\rangle$. On the other hand, we use the standard Euclidean scalar product $\langle x, x\rangle$. So our result is slightly different from the result of Rangachari. Indeed, if we denote by $\bar{\Theta}_{A_{p-1}^*}(\tau)$ the theta function defined by Rangachari, we have $\bar{\Theta}_{A_{p-1}^*}(\tau) = \Theta_{A_{p-1}^*}(p\tau)$.

2. Preliminaries. A lattice in $\boldsymbol{R}^n$ is a free $\boldsymbol{Z}$-submodule $L = \boldsymbol{Z}e_1 \oplus \cdots \oplus \boldsymbol{Z}e_n$ of $\boldsymbol{R}^n$ with a $\boldsymbol{R}$-independent $\boldsymbol{Z}$-basis $\{e_1, \ldots, e_n\}$. We use the standard Euclidean scalar product $\langle x, y\rangle = \sum x_i y_i$ of $x, y \in \boldsymbol{R}^n$.

Let L be a lattice with basis $\{e_1, \ldots, e_n\}$, and let A be the Gram matrix $(\langle e_i, e_j\rangle)$. Then we define the determinant $\det L$ of L by the deter-

*) This paper is a part of author's master thesis at Tohoku University.

minant $\det A$ of the Gram matrix A: $\det L = \det(\langle e_i, e_j\rangle)$. One can easily see that $\det L$ does not depend on the choice of the basis.

Let L be a lattice in $\boldsymbol{R}^n$. We denote the dual lattice of L by L^*:

$$L^* = \{x \in \boldsymbol{R}^n \mid \langle x, y\rangle \in \boldsymbol{Z} \text{ for all } y \in L\}.$$

The theta function $\Theta_L(\tau)$ of the lattice L is defined by

$$\Theta_L(\tau) = \sum_{x\in L} \exp(\pi i\tau\langle x, x\rangle)$$

for $\tau \in \boldsymbol{H}$, and satisfies the following transformation formula (cf. e.g. [5]):

Proposition 2.1. In choosing suitably the sign of $\left(\frac{\tau}{i}\right)^{n/2}$, we have

$$\Theta_L\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{n/2} \frac{1}{\sqrt{\det L}}\, \Theta_{L^*}(\tau).$$

For $n > 1$, let

$$A_{n-1} = \{x \in \boldsymbol{Z}^n \mid x_1 + \cdots + x_n = 0\}.$$

A_{n-1} lies in the hyperplane $x_1 + \cdots + x_n = 0$ in $\boldsymbol{R}^n$, and is a lattice of rank $n-1$.

Let m be an integer > 1, and let R be the finite ring $\boldsymbol{Z}/m\boldsymbol{Z}$. Then a code of length n over R is a nonempty subset of R^n, and a linear code of length n over R is a R-submodule of R^n. For any linear code C of length n over R, we define the dual code C^* of C by

$$C^* = \{x \in R^n \mid \langle x, y\rangle = 0 \text{ for all } y \in C\},$$

where $\langle x, y\rangle$ is the standard scalar product $\sum x_i y_i$.

For m odd, set $\omega = (m-1)/2$; for m even, set $\omega = m/2$. Then any element of R can be represented by an integer of absolute value $\leq \omega$. Hence $R = \{\pm i\}_{0\leq i\leq\omega}$, where we assume $\pm 0 = 0$, and for m even $\pm\omega = \omega$. Then the Lee weight enumerator $P_C(X_0, \ldots, X_\omega)$ of a linear code $C \subset R^n$ is defined by

$$P_C(X_0, \ldots, X_\omega) = \sum_{x\in C} X_\omega^{\sigma_0(x)} \cdots X_\omega^{\sigma_\omega(x)},$$

where $\sigma_i(x)$ is the number of $+i$ or $-i$ occurring in the coordinate $(x_1, \ldots, x_n)$ of $x \in R^n$. $P_C(X_0, \ldots, X_\omega)$ is a homogeneous polynomial of $X_0, \ldots, X_\omega$ of degree n.

Take the standard lattice $\boldsymbol{Z}^n \subset \boldsymbol{R}^n$ and consider the reduction map mod m

$$\rho : \boldsymbol{Z}^n \longrightarrow (\boldsymbol{Z}/m\boldsymbol{Z})^n = R^n.$$

For any linear code C of length n over R, we define the associated lattice $L(C)$ by

$$L(C) = \frac{1}{\sqrt{m}}\rho^{-1}(C).$$

3. Result. Theorem 3.1. *If* $\varphi_{m,l}(\tau) = \vartheta_{l/m,0}(0, m\tau)$, *then*

$$P_C(\varphi_{m,0}(\tau), \ldots, \varphi_{m,\omega}(\tau)) = \Theta_{L(C)}(\tau).$$

Proof. $x \in L(C)$ if and only if $x = (1/\sqrt{m})(w+y)$, $(w \in C,\ y \in m\boldsymbol{Z}^n)$. Since

$$\varphi_{m,l}(\tau) = \sum_{y\in Z} \exp\left\{\pi i\left(y + \frac{l}{m}\right)^2 m\tau\right\} = \sum_{y\in mZ} \exp\left\{\frac{\pi i\tau}{m}(y+l)^2\right\},$$

we have

$$\begin{aligned}\Theta_{L(C)}(\tau) &= \sum_{x\in L(C)} \exp(\pi i\tau\|x\|^2)\\ &= \sum_{w\in C}\sum_{y\in mZ^n} \exp\left(\frac{\pi i\tau}{m}\|w+y\|^2\right)\\ &= \sum_{w\in C}\sum_{y\in mZ^n}\prod_{j=1}^{n} \exp\left\{\frac{\pi i\tau}{m}(w_j+y_j)^2\right\}\\ &= \sum_{w\in C}\prod_{j=1}^{n}\sum_{y\in mZ^n} \exp\left\{\frac{\pi i\tau}{m}(w_j+y_j)^2\right\}\\ &= \sum_{w\in C} \varphi_{m,0}(\tau)^{\sigma_0(w)}\varphi_{m,1}(\tau)^{\sigma_1(w)}\cdots\varphi_{m,\omega}(\tau)^{\sigma_\omega(w)}.\end{aligned}$$

□

We now consider in particular the repetition code $C = \{(i, i, \ldots, i)\}_{|i|\leq\omega}$ of length m. Then the weight enumerator of C is given by

$$P_C(X_0, \ldots, X_\omega) = \begin{cases} X_0^m + 2\sum_{i=1}^{\omega} X_i^m, & m \text{ odd},\\ X_0^m + 2\sum_{i=1}^{\omega} X_i^m - X_\omega^m, & m \text{ even}.\end{cases}$$

It follows from the above theorem, that

$$(3.1)\quad \Theta_{L(C)}(\tau) = \begin{cases}\vartheta_{0,0}(0, m\tau)^m + 2\sum_{l=1}^{\omega}\vartheta_{l/m,0}(0, m\tau)^m, & m \text{ odd},\\ \vartheta_{0,0}(0, m\tau)^m + 2\sum_{l=1}^{\omega}\vartheta_{l/m,0}(0, m\tau)^m\\ \qquad - \vartheta_{\omega/m,0}(0, m\tau)^m, & m \text{ even}.\end{cases}$$

We put $z = 0$ in the Ramanujan identity. Since $\vartheta_{r/n,0}(0, n\tau) = \vartheta_{-r/n,0}(0, n\tau)$, the right side of (3.1) is precisely $\vartheta_{0,0}(0, \tau)\cdot F(\tau)$. Hence $\Theta_{L(C)}(\tau) = \vartheta_{0,0}(0, \tau)\cdot F(\tau)$.

Lemma 3.2.

$$L(C)^* = \frac{1}{\sqrt{m}}A_{m-1} \oplus \frac{1}{\sqrt{m}}\boldsymbol{Z}\overbrace{(1, \ldots, 1)}^{m \text{ times}}.$$

Proof.

$$L(C) = \frac{1}{\sqrt{m}}\{(x_1, \ldots, x_m) \in \boldsymbol{Z}^m \mid x_1 \equiv \cdots \equiv x_m \pmod{m}\}.$$

So $y/\sqrt{m} \in L(C)^*$ if and only if $\langle x, y\rangle = j\sum_{i=1}^m y_i + m\sum_{i=1}^m t_i y_i \in m\boldsymbol{Z}$ for all $x/\sqrt{m} \in L(C)$, where $x_i = j + mt_i$ $(j = 0, \ldots, m-1,\ t_i \in \boldsymbol{Z})$. Here the first term satisfies $y_1 + \cdots + y_m \equiv 0 \pmod{m}$ and the second term satisfies $y_i \in \boldsymbol{Z}$. Hence

$$L(C)^* = (1/\sqrt{m})\,\{(y_1, \ldots, y_m) \in \boldsymbol{Z}^m \mid y_1$$

$+ \cdots + y_m \equiv 0 \pmod{m}\}$.

For any $y/\sqrt{m} \in L(C)^*$, we have $y_1 + \cdots + y_m = tm$ with a certain $t \in \boldsymbol{Z}$. Hence $y = (y_1 - t, \ldots, y_m - t) + t(1, \ldots, 1)$. Therefore we have $L(C)^* = (1/\sqrt{m})A_{m-1} \oplus (1/\sqrt{m})\boldsymbol{Z}(1, \ldots, 1)$. □

By using Proposition 2.1 and Lemma 3.2, we can prove Theorem 1.1.

Proof of Theorem 1.1. Let the notation be as before. Then we have

$$\begin{aligned}
&\boldsymbol{\Theta}_{L(C)}\left(-\frac{1}{\tau}\right) \\
&= \left(\frac{\tau}{i}\right)^{m/2} \frac{1}{\sqrt{\det L(C)}} \Theta_{L(C)^*}(\tau) \\
&= \left(\frac{\tau}{i}\right)^{m/2} \frac{1}{\sqrt{\det L(C)}} \Theta_{(1/\sqrt{m})A_{m-1} \oplus (1/\sqrt{m})Z(1,1,\ldots,1)}(\tau) \\
&= \left(\frac{\tau}{i}\right)^{m/2} \frac{1}{\sqrt{\det L(C)}} \boldsymbol{\Theta}_{(1/\sqrt{m})A_{m-1}}(\tau) \cdot \Theta_{(1/\sqrt{m})Z(1,1,\ldots,1)}(\tau) \\
&= \left(\frac{\tau}{i}\right)^{m/2} \frac{1}{\sqrt{\det L(C)}} \boldsymbol{\Theta}_{(1/\sqrt{m})A_{m-1}}(\tau) \cdot \Theta_Z(\tau).
\end{aligned}$$

By substituting $-1/\tau$ in τ, we obtain

$$\begin{aligned}
&\Theta_{L(C)}(\tau) \\
&= \left(\frac{-1}{i\tau}\right)^{m/2} \frac{1}{\sqrt{\det L(C)}} \Theta_{(1/\sqrt{m})A_{m-1}}\left(-\frac{1}{\tau}\right) \cdot \Theta_Z\left(-\frac{1}{\tau}\right) \\
&= \left(\frac{-1}{i\tau}\right)^{m/2} \left(\frac{\tau}{i}\right)^{(m-1)/2} \left(\frac{\tau}{i}\right)^{1/2} \frac{1}{\sqrt{\det L(C)}} \frac{1}{\sqrt{\det(1/\sqrt{m})A_{m-1}}} \\
&\quad \times \Theta_{\sqrt{m}A^*_{m-1}}(\tau) \cdot \Theta_Z(\tau) \\
&= \Theta_{\sqrt{m}A^*_{m-1}}(\tau) \cdot \Theta_Z(\tau) \\
&= \Theta_{A^*_{m-1}}(m\tau) \cdot \vartheta_{0,0}(0, \tau).
\end{aligned}$$

(Note that $\det L(C) \cdot \det (1/\sqrt{m})A_{m-1} = 1$ follows from Lemma 3.2.)

Since $\Theta_{L(C)}(\tau) = \vartheta_{0,0}(0, \tau) \cdot F(\tau)$, we obtain $F(\tau) = \Theta_{A^*_{m-1}}(m\tau)$.

Remark 3.3. (1) Using the standard expression (see [1, p. 110 and p. 115])

$$\Theta_{A^*_{m-1}}(\tau) = \frac{1}{m} \sum_{l=0}^{m-1} \left\{ \sum_{k=0}^{m-1} \zeta^{-kl} \frac{\vartheta_{0,k/m}(0, \tau)^m}{\vartheta_{l/m,0}(0, m\tau)} \right\},$$

where $\zeta = \exp(2\pi i/m)$, we can express the Ramanujan identity as an identity involving only Jacobi theta functions. Substituting this expression of $\Theta_{A^*_{m-1}}(\tau)$ in the Ramanujan identity, we obtain the following result: For m odd,

$$\sum_{l=-(m-1)/2}^{(m-1)/2} \frac{\vartheta_{l/m,0}(z, \tau)^m}{\vartheta_{0,0}(z, m^{-1}\tau)} = \frac{1}{m} \sum_{k,l=0}^{m-1} \zeta^{-kl} \frac{\vartheta_{0,k/m}(0, \tau)^m}{\vartheta_{l/m,0}(0, m\tau)}.$$

For m even,

$$\sum_{l=-m/2+1}^{m/2} \frac{\vartheta_{l/m,0}(z, \tau)^m}{\vartheta_{0,0}(z, m^{-1}\tau)} = \frac{1}{m} \sum_{k,l=0}^{m-1} \zeta^{-kl} \frac{\vartheta_{0,k/m}(0, \tau)^m}{\vartheta_{l/m,0}(0, m\tau)}.$$

(2) There is a doubt on the validity of second formula of [4, p. 371]. For, as has been mentioned at Remark 1.2, Rangachari defined the theta function by using Voronoi's principal form. But the standard expression is defined by using the standard Euclidean scalar product. It seems that he confused two kinds of theta functions with different definitions.

Acknowledgments. The author is deeply grateful to Prof. Yasuo Morita and Prof. Koichi Takase for their constant encouragement and valuable discussions during the preparation of this paper.

References

[1] J. H. Conway and N. J. A. Sloane: Sphere Packings, Lattices and Groups. Springer-Verlag, New York/Berlin (1988).

[2] D. P. Maher: Lee polynomials of codes and theta functions of lattices. Canad. J. Math., **30**, 738–747 (1978).

[3] S. Ramanujan: The Lost Notebook and Other Unpublished Papers. Narosa Publishing House, New Delhi (1988).

[4] S. S. Rangachari: On a result of Ramanujan on theta functions. J. Number Theory, **48**, 364–372 (1994).

[5] J. P. Serre: A Course in Arithmetic. Springer, New York etc. (1973).