Invariance of strong paracompactness under closed-and-open maps

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Abstract: In this paper we strengthen a result of Ponomarev [6] on the invariance of Strong Paracompactness under open perfect maps. Namely, we prove that Strong Paracompactness is invariant under closed-and-open maps. This result follows after we give four new characterizations of strong paracompactness.

Key words: Strong paracompactness; closed-and-open maps.

§1 Preliminaries. In this paper by a space we mean a Hausdorff topological space and by a map, a continuous map of spaces.

Strongly paracompact spaces were defined by Dowker [2]. Unlike most covering properties, strongly paracompact spaces are not invariant under perfect maps [1]. This can be seen from the following result by Ponomarev [5].

Theorem 1.1. *Every paracompact space is the image of a strongly paracompact space under a perfect map.* □

Ponomarev [6] has shown that strong paracompactness is an invariant of open perfect maps. Indeed, Ponomarev [7] strengthened this result by showing that strong paracompactness is preserved under Λ-maps in the realm of regular spaces, where a map $f: X \rightarrow Y$ is called an Λ-map if it satisfies the following two conditions : (a) the image of every clopen (closed-and-open) set is a clopen set and (b) for every point $y \in Y$ and every cover $\mathcal{U}_y$ of the set $f^{-1}y$ by clopen sets admits a finite subcover. The aim of this paper is to show that in fact in the realm of regular spaces, strong paracompactness is preserved under CO-maps, that is maps satisfying condition (a) above. As a consequence we get that strong paracompactness is an invariant of closed-and-open maps (i.e. maps which are both closed and open).

Let $\mathcal{P} = \{P_\alpha : \alpha \in \mathcal{A}\}$ be a collection of subsets of a set X. By a *chain* from P_α to $P_{\alpha'}$ we mean a finite sequence $P_{\alpha(1)}, P_{\alpha(2)}, \ldots, P_{\alpha(k)}$ of elements of $\mathcal{P}$ such that $\alpha(1) = \alpha$, $\alpha(k) = \alpha'$ and $P_{\alpha(i)} \cap P_{\alpha(i+1)} \neq \emptyset$ for $i = 1, \ldots, k-1$. The collection $\mathcal{P}$ is said to be *connected* if for every pair P_α, $P_{\alpha'}$ of elements of $\mathcal{P}$ there exists a chain from P_α to $P_{\alpha'}$. For every collection $\mathcal{P}$ the *components* of $\mathcal{P}$ are defined as maximal connected subcollections of $\mathcal{P}$, that is connected subcollections of $\mathcal{P}$ which are not proper subsets of any connected subcollection of $\mathcal{P}$.

Remember that a collection $\mathcal{P}$ of subsets of a set X is said to be star-finite (star-countable) if for every $P \in \mathcal{P}$ the collection $\{Q \in \mathcal{P} : Q \cap P \neq \emptyset\}$ is finite (countable). A space X is called *strongly paracompact* (otherwise called *hypocompact*) if every open cover of X has a star-finite open refinement. Thus every strongly paracompact space is paracompact but the converse is not true. In fact there exist metric spaces which are not strongly paracompact [1].

The following lemma will be used below (see for example [1] or [3]).

Lemma 1.2.

(1) *Every collection $\mathcal{P}$ of subsets of a set X decomposes into the union of its components.*

(2) *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct components of $\mathcal{P}$, then $(\cup \mathcal{P}_1) \cap (\cup \mathcal{P}_2) = \emptyset$.*

(3) *If $\mathcal{P}$ is star-countable, then each component is a countable subcollection of $\mathcal{P}$.* □

For a collection $\mathcal{P}$ of subsets of a set X and an infinite ordinal number τ let $\mathcal{P}^\tau = \{\cup \mathcal{Q} : \mathcal{Q} \subset \mathcal{P}, |\mathcal{Q}| < \tau\}$. The collection $\mathcal{P}^\omega = \mathcal{P}^{\omega_0}$ is usually denoted by $\mathcal{P}^F$. In this paper we will be interested in the particular case of $\tau = \omega_1$, the first uncountable ordinal. Thus $\mathcal{P}^{\omega_1}$ is the collection of all unions of at most countable subcollections from $\mathcal{P}$.

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§2 Characterizations of strongly paracompact spaces. We now turn to the known characterizations of strong paracompactness.

Theorem 2.1. *For every regular space X the following conditions are equivalent*:

(i) *The space X is strongly paracompact*;

(ii) *Every open cover of the space X has a closed refinement which is both locally finite and star-finite*;

(iii) *Every open cover of the space X has a closed refinement which is both locally finite and star-countable*;

(iv) *Every open cover of the space X has a star-countable refinement.* □

This theorem was proved by Smirnov [8]. One can state the following result as a corollary (which was proved earlier by Morita [4]).

Corollary 2.2. *Every Lindelöf space is strongly paracompact.* □

We now prove other characterizations for strongly paracompact spaces.

Theorem 2.3. *For every regular space X the following conditions are equivalent*:

(i) *The space X is strongly paracompact*;

(ii) *For every open cover $\mathscr{U}$ of the space X, $\mathscr{U}^{\omega_1}$ has a disjoint open refinement* (*i.e. an open refinement of order* 0);

(iii) *For every open cover $\mathscr{U}$ of the space X, $\mathscr{U}^{\omega_1}$ has a closure preserving clopen refinement*;

(iv) *For every open cover $\mathscr{U}$ of the space X, $\mathscr{U}^{\omega_1}$ has a σ-closure preserving clopen refinement*;

(v) *For every open cover $\mathscr{U}$ of the space X, $\mathscr{U}^{\omega_1}$ has a σ-discrete clopen refinement.*

Proof. We start with (i) $\Rightarrow$ (ii). Let the space X be strongly paracompact and let $\mathscr{U}$ be an open cover of X. Then by definition, the cover $\mathscr{U}$ has a star-finite open refinement $\mathscr{V}$. It is not difficult to see that (ii) follows after applying Lemma 1.2 to the cover $\mathscr{V}$.

The implication (ii) $\Rightarrow$ (iii) follows from the trivial fact that any disjoint open cover is closure preserving and the implication (iii) $\Rightarrow$ (iv) is evident.

We now prove the implications (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (ii). Let the space X satisfy condition (iv) and let $\mathscr{U}$ be an open cover of X. Then the cover $\mathscr{U}^{\omega_1}$ has a σ-closure preserving clopen refinement $\mathscr{F} = \bigcup_{n=1}^{\infty}\mathscr{F}_n$, where each $\mathscr{F}_n$ is a closure preserving clopen collection. Let $\mathscr{F}_n = \{F_\alpha : \alpha < \gamma_n\}$ be a well-ordering of $\mathscr{F}_n$ for every $n \in \boldsymbol{N}$. For every $F_\alpha \in \mathscr{F}_n$ let $W_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$ and for every $n \in \boldsymbol{N}$ let $\mathscr{W}_n = \{W_\alpha : F_\alpha \in \mathscr{F}_n\}$. It is not difficult to see that each $\mathscr{W}_n$ is a discrete clopen collection and so $\mathscr{W} = \bigcup_{n=1}^{\infty}\mathscr{W}_n$ is a σ-discrete clopen refinement of $\mathscr{U}^{\omega_1}$. Thus condition (v) is satisfied. Let $P_n = \bigcup \mathscr{W}_n = \bigcup \{W_\alpha : W_\alpha \in \mathscr{W}_n\}$ for every $n \in \boldsymbol{N}$. Then P_n is a clopen set and therefore $Q_n = \bigcup_{k=1}^{n} P_k$ is also clopen for every $n \in \boldsymbol{N}$. Let $\mathscr{Q}_1 = \mathscr{W}_1$ and $\mathscr{Q}_{n+1} = \{W \setminus Q_n : W \in \mathscr{W}_{n+1}\}$ for every n $\in \boldsymbol{N}$. The collection $\mathscr{Q} = \bigcup_{n=1}^{\infty}\mathscr{Q}_n$ is a disjoint open refinement of $\mathscr{U}^{\omega_1}$ and thus (ii) is satisfied.

Finally we show the implication (ii) $\Rightarrow$ (i). Let the space X satisfy condition (ii) and let $\mathscr{U}$ be an open cover of X. Then the cover $\mathscr{U}^{\omega_1}$ has a disjoint open refinement $\mathscr{V}$. For every $V \in \mathscr{V}$ choose an element $G(V) \in \mathscr{U}^{\omega_1}$ such that $V \subset G(V)$. By definition one can choose a countable collection $\mathscr{U}(V) = \{U_n : U_n \in \mathscr{U}, n \in \boldsymbol{N}\}$ such that $G(V) = \bigcup \mathscr{U}(V)$. Let $\mathscr{W}(V) = V \wedge \mathscr{U}(V) = \{V \cap U_n : U_n \in \mathscr{U}(V)\}$. The cover $\mathscr{W} = \bigcup \{\mathscr{W}(V) : V \in \mathscr{V}\}$ is easily seen to be a star-countable open refinement of the cover $\mathscr{U}$ and thus (i) follows from condition (iv) of Theorem 2.1. □

§3 CO-Maps and strong paracompactness. In this section we prove our main result concerning the invariance of strong paracompactness under closed-and-open maps (CO-maps for regular spaces).

Theorem 3.1. *Let f be a CO-map of a strongly paracompact space X onto a regular space Y. Then the space Y is also strongly paracompact.*

Proof. Let $\mathscr{U}$ be an open cover of the space Y. Then $f^{-1}\mathscr{U}$ is an open cover of X and so there exists a closure preserving clopen refinement $\mathscr{V}$ of $(f^{-1}\mathscr{U})^{\omega_1}$. Since the map f is a CO-map we have that the cover $f\mathscr{V}$ of Y is a clopen refinement of $\mathscr{U}^{\omega_1}$. For every subcollection $\mathscr{V}' \subset \mathscr{V}$ we have $f(\bigcup\mathscr{V}') = \bigcup f\mathscr{V}'$ and so the clopen cover $f\mathscr{V}$ is closure preserving. This proves that the space Y is strongly paracompact by Theorem 2.2 (iii). □

Finally, since regularity is an invariant of closed-and-open maps we have.

Corollary 3.2. *Let f be a closed-and-open map of a strongly paracompact space X onto a space Y. Then Y is also strongly paracompact.* □

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