A resolvent estimate and a smoothing property of inhomogeneous Schrödinger equations

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1. Results. Throughout this paper, we always assume \( n \geq 2 \). Let \( \rho(\xi) > 0 \) be of the class \( C^\infty(R^n \setminus 0) \) and positively homogeneous of degree 1, and \( P = \rho(D_x) = \mathcal{F}^{-1}_\xi \rho(\xi) \mathcal{F}_x \) the corresponding Fourier multiplier. Suppose that \( \Sigma = \{ \xi; \rho(\xi) = 1 \} \) has non-vanishing Gaussian curvature. The objective of this brief article is to show the following smoothing effect of inhomogeneous generalized Schrödinger equations:

**Theorem 1.1.** Suppose \( 1 - n/2 < s < 1/2 \), \( 1 - n/2 < \alpha < 1/2 \) and let \( |x|^{s-\alpha} f(t, x) \in L^2(R_+ \times R^n) \). Then there exists a unique solution \( u(t,x) \) to

\[
(\partial_t + iP) u = f
\]

which satisfies \( |x|^{a-1} |D_x|^{s+a} u(t, x) \in L^2(R_+ \times R^n) \).

Theorem 1.1 says that the solution gains the regularity of order \( "s" \) in connection with the decay order of the inhomogeneous term \( f \), plus an extra gain of order \( "\alpha < 1/2" \), in the sense of space-time norm. This is an improvement of the result in Hoshiro [3] which showed Theorem 1.1 with \( P = D_z \) and \( 0 < \alpha = s < 1/2 \).

Since Hoshiro’s method deeply depends on the properties of special functions, it is not suitable for handling the general operator \( P \). To remove this obstacle is also in our focus. The most essential part of the proof is the following resolvent estimate:

**Theorem 1.2.** Suppose \( 1 - n/2 < a < 1/2 \) and \( 1 - n/2 < b < 1/2 \). Then we have

\[
\sup_{\lambda \geq 0} \left| x^{a-1} D_x^{a+b} (P^2 - \lambda^2)^{-1} v(x) \right|_{L^2(R^n)} \leq C \left| x^{a-b} v(x) \right|_{L^2(R^n)}.
\]

Theorem 1.2 is partly proved in the master’s thesis of the second author [7]. The main tools for the proof of it are the weighted \( L^2 \)-boundedness of Fourier multipliers, the limiting absorption principle, and an estimate for the kernel of the resolvent, which enable us to treat general operators \( P \). We shall explain the details in Section 2.

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2. Proof. To begin with, we shall prove Theorem 1.2. The argument here is based on [7]. Hereafter, we denote the norm \( \| \cdot \|_{L^2(R^n)} \) by \( \| \cdot \| \).

We remark

\[
1/2 < 1 - a < n/2, \quad 1/2 < 1 - b < n/2, \quad 0 < a + b - 2 + n < n,
\]

which will be used later frequently without any notice. Furthermore, we may assume \( (3 - n)/2 \leq a + b \).

The general case can be reduced to this special one because of the following:

**Proposition 2.1** ([5, Theorem B*]). Suppose \( k < n/2, \quad l < n/2, \quad 0 < m < n, \quad k + l + m = n \). Then we have

\[
|v(y)| \leq C \left| x^{a-b} v(x) \right|_{L^2(R^n)}.
\]

In fact, if \( a + b < (3 - n)/2 \), we have

\[
(3 - n)/2 \leq (a + \delta) + b \quad \text{and} \quad 1 - n/2 < (a + \delta) < 1/2,
\]

where \( \delta = (3 - n)/2 - (a + b) \). We remark \( 0 < \delta < (n - 1)/2 \). Then, by Proposition 2.1 and the estimate (1.2) with \( a \) replaced by \( a + \delta \), we have

\[
\sup_{\lambda \geq 0} \left| x^{a-1} D_x^{a+b} (P^2 - \lambda^2)^{-1} v \right|_{L^2(R^n)} \leq C \sup_{\lambda \geq 0} \left| x^{(a+\delta)-1} D_x^{a+\delta+b} (P^2 - \lambda^2)^{-1} v \right|_{L^2(R^n)} \leq C \left| x^{a-b} v \right|_{L^2(R^n)}.
\]
which is the estimate (1.2).

Now, all we have to show is, by the scaling argument, the following two estimates:

\[ (2.1) \sup_{|m| \geq 0} \left| x \right|^{a+b} |D|^a \left( P^2 - \chi^2 \right)^{-1} (1 - \varphi \circ p)(D)(D) v \leq C \left| x \right|^{1-b} \| v \|, \]

\[ (2.2) \sup_{|m| \geq 0} \left| x \right|^{a+b} |D|^a \left( P^2 - \chi^2 \right)^{-1} \left( \varphi \circ p \right)(D)(D) v \leq C \left| x \right|^{1-b} \| v \|, \]

where \( \varphi (\rho) \in \mathcal{C}_c^\infty (\mathbb{R}_+) \) is a function which is equal to 1 near \( \rho = 1 \).

The estimate (2.1) is a consequence of Proposition 2.1 and the following:

**Proposition 2.2** ([6, Chapter 11, Theorem 5]).

Suppose \(-n/2 < k < n/2\). Then we have

\[ \left| x \right|^b \left| \mathcal{D}^{m} (D) \mathcal{V} \right| \leq C \sum_{|\xi| \leq N} \sup_{|\xi| \leq N} \left| \xi \right|^{a+b} \left| \mathcal{D}^{m} (D) \mathcal{V} \right| \left| x \right|^b \mathcal{V}. \]

In fact, setting \( m_\xi (\xi) = \left| \xi \right|^2 (p(\xi)^2 - \chi^2)^{-1} (1 - \varphi \circ p)(\xi) \), we have

\[ \sup_{|m| \geq 0} \left| x \right|^{a+b} |D|^a \left( P^2 - \chi^2 \right)^{-1} m_\xi (D) v \leq C \sup_{|m| \geq 0} \left| x \right|^{1-b} m_\xi (D) v \leq C \left| x \right|^{1-b} \| v \|, \]

which is the estimate (2.1).

The estimate (2.2) is easily obtained from Proposition 2.2 and the estimate

\[ (2.3) \sup_{|m| \geq 0} \left| x \right|^{a+b} P^{a+b} \left( P^2 - \chi^2 \right)^{-1} \left( \varphi \circ p \right)(D)(D) v \leq C \left| x \right|^{1-b} \| v \|, \]

which is a consequence of the following two propositions: (The curvature condition of \( \Sigma \) is necessary for Proposition 2.4 only).

**Proposition 2.3** ([1, Theorem 14.2.2]).

Let \( \mathcal{V} \in \mathcal{C}_c^\infty (\mathbb{R}_+) \). Suppose \( k > 1/2 \) and \( l > 1/2 \). Then we have

\[ \sup_{|m| \geq 0} \left| x \right|^{b} \left| \mathcal{D}^{m} (D) \mathcal{V} \right| \leq C \left| x \right|^{1-b} \mathcal{V} \leq C \left| x \right|^{1-b} \mathcal{V}. \]

**Proposition 2.4** ([4, Theorem 6.3]).

Let \( \varphi \in \mathcal{C}_c^\infty (\mathbb{R}_+) \). Then we have

\[ \left| \mathcal{F}^{-1} \left[ \left( \varphi (\xi)^2 - \chi^2 \right)^{-1} \left( \varphi \circ p \right)(\xi) \right] (x) \right| \leq \frac{C}{x^n} \left( P^2 - \chi^2 \right)^{-1} \varphi (\xi)^2 \left( \varphi \circ p \right)(\xi) \]

\[ \leq C \left| x \right|^{-n/2}. \]

In fact, setting \( \varphi (\rho) = \rho^{a+b} \varphi (\rho) \) and \( \mathcal{V} = \mathcal{V} \varphi \), we have

\[ \sup_{|m| \geq 0} \left| x \right|^{b} \left| \mathcal{D}^{m} (D) \mathcal{V} \right| \leq C \left| x \right|^{1-b} \mathcal{V} \]

by Proposition 2.3, where \( \chi(x) \) is the characteristic function of the set \( \{ x \mid \left| x \right| \leq 1 \} \). On the other hand, since \((3 - n)/2 \leq a + b \) implies \( 1 - a \leq b + (n - 1)/2 < n/2 \), we have

\[ \sup_{|m| \geq 0} \left| x \right|^{b} \left| \mathcal{D}^{m} (D) \mathcal{V} \right| \leq C \left| x \right|^{1-b} \mathcal{V}. \]

Thus we have obtained the estimate (2.3) and completed the proof of Theorem 1.2.

As is also explained in Hoshiro [21 and [3], we can construct the solution \( u \) to the inhomogeneous equation (1.1) by taking the weak limit of the functions

\[ u_\pm (t, x) = \frac{1}{i} \mathcal{F}^{-1} (p^2 + (\tau - i\varepsilon) \mathcal{F} f_\pm (t, x) \]

\[ + \frac{1}{i} \mathcal{F}^{-1} (p^2 + (\tau + i\varepsilon) \mathcal{F} f_\pm (t, x) \]

as \( \varepsilon \to 0 \) in an appropriate function spaces. Here \( f_\pm \) denote the function \( f \) multiplied by the characteristic function of the set \( \{ t \mid \pm t \geq 0 \} \). By Theorem 1.2 with \( a = \alpha, b = s \), this argument can be justified, and we have Theorem 1.1.

References


