Exotic group actions in dimension four and Seiberg-Witten theory

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Topology of smooth 4-manifolds has been studied extensively by Donaldson and Seiberg-Witten theory. In [10] we used Donaldson invariants of degree 0 to give examples of exotic free actions of certain finite groups in dimension 4. In this paper we will generalize the result in [10] by Seiberg-Witten theory. We discuss Donaldson and Seiberg-Witten invariants for connected sums of 4-manifolds and rational homology 4-spheres in §1 according to [11]. In §2 by the constructions similar to those in [10] together with Cooper-Long's result [1] we show

Theorem. *For any nontrivial finite group G there exists a 4-manifold that has infinitely many free G actions so that their orbit spaces are homeomorphic but mutually non-diffeomorphic.*

§1. Invariants for some reducible manifolds. Let us recall the definitions of Donaldson and Seiberg-Witten invariants briefly. See [2], [6], [8], [12] for details. Let X be a closed smooth oriented 4-manifold with $b_1(X) = 0$, $b_2^+(X) > 1$ and let P be a principal $SO(3)$ bundle over X with $w_2(P) \equiv w \pmod 2$ for some $w \in H^2(X, \mathbf{Z})$ (and hence P is a reduction of a $U(2)$ bundle $\tilde{P}$). Hereafter $w \pmod 2$ is denoted simply by w. Let $\mathcal{G}_P$ be the set of automorphisms of P covered by those of $\tilde{P}$ with $\det = 1$. Define $\mathcal{M}_P$ to be the space of ASD (anti-self-dual) connections modulo $\mathcal{G}_P$ with respect to a generic metric on X. Then for the symmetric product $z = x^t v_1 \cdots v_s$ with the generator x of $H_0(X)$ and $v_i \in H_2(X)$, there exists a subspace $\mathcal{M}_P \cap V_z$ of codimension $4t + 2s$ in $\mathcal{M}_P$ such that the Donaldson invariant $D_X^w(z)$ is defined by the number of points in $\mathcal{M}_P \cap V_z$ counted with sign for a bundle P with $w_2(P) \equiv w$ and $-2p_1(P) - 3(1 + b_2^+(X)) = 4t + 2s$ (put $D_X^w(z) = 0$ if there does not exist such a bundle). Here note that if there are no flat connections on any $SO(3)$ bundle over X with $w_2 \equiv w$ then $\mathcal{M}_P \cap V_z$ is compact ([6]). Otherwise to avoid the flat connections we replace (X, P) by $(X \# \overline{CP}^2, P \# Q)$, where Q is the reducible $SO(3)$ bundle over $\overline{CP}^2$ with w_2 being the Poincare dual of the generator z_0 of $H_2(\overline{CP}^2, \mathbf{Z})$ modulo 2, and replace $D_X^w(z)$ by $D_{X\#\overline{CP}^2}^{w+z_0}(zz_0)$ (Morgan-Mrowka trick, [6]). In Seiberg-Witten theory, we consider a spin^c structure c on X, the associated $\pm$ spinor bundle $W^{\pm}$, and its determinant complex line bundle L over X. Then the Seiberg-Witten moduli space $\mathcal{M}_X(c)$ is the space of pairs of connections A on L and cross sections ϕ of W^+ satisfying the Seiberg-Witten equation modulo $\mathrm{Map}(X, S^1)$.

$$(SW)\quad \mathcal{D}_A(\phi) = 0,\ F^+(A) + \delta = (\phi^* \otimes \phi)_0$$

(see [8], [12] for the definitions.) The space $\mathcal{M}_X(c)$ is a compact oriented manifold of dimension $d(L) = (c_1(L)^2 - 2\chi - 3\sigma)/4$ for a generic metric on X where χ and σ are the euler number and the signature of X. Hereafter $c_1(L)$ is denoted simply by L. The Seiberg-Witten (SW) invariants $SW_X(L)$ for L with $d(L) = 0$ is the sum of the numbers of points in $\mathcal{M}_X(c)$ counted with sign for all spin^c structures c corresponding to L. (see [8] for the definition in case $d(L) > 0$.) L is called a Seiberg-Witten (SW) class if $SW_X(L) \neq 0$. X is called SW simple if $SW_X(L) = 0$ whenever $d(L) > 0$. Hereafter we assume that $H_1(X, \mathbf{Z}) = 0$, $b_2^+(X) > 1$, and Y is a rational homology 4-sphere. Moreover we assume that X is SW simple and KM simple, that is, $D_X^w(x^2 z) = 4D_X^w(z)$ for any $w \in H^2(X, \mathbf{Z})$, $z \in Sym\,(H_0(X) \oplus H_2(X))$, and satisfies the following equation discussed in [12].

$$(W)\quad D_X^w((1 + x/2)e^v) = 2^{2+(7\chi+11\sigma)/4} e^{Q/2} \sum (-1)^{(w^2+wL)/2} SW_X(L) e^L(v)$$

where $v \in H_2(X)$, Q is the intersection form of X, and the sum on the right hand side is taken over all the SW classes L of X.

The following results about these invariants for $X \# Y$ may be known to the experts, but we cannot find them in explicit forms in the literature.

Proposition 1.1 [11]. *If X satisfies the above* *conditions, then so does* $X\#Y$. *For any* $v \in H_2(X, \mathbf{R}) \cong H_2(X\#Y, \mathbf{R})$ *and for any* $w + w' \in H^2(X, \mathbf{Z}) \oplus H^2(Y, \mathbf{Z})$, *the both sides of* (W) *for* $X\#Y$ *are* $|H_1(Y, \mathbf{Z})|$ *times those of* (W) *for* X, v, *and* w.

Proposition 1.2 [11]. (1) *For each* $w' \in H^2(Y, \mathbf{Z})$ *with* $w' \equiv w_2(Y)$ (mod2) *there exists a complex line bundle* L' *over* Y *with* $c_1L' = w'$ *and the set of SW classes of* $X\#Y$ *is given by* $\{L + L' \mid L$ *is a SW class for* X, $L' \equiv w_2(Y)$ (mod2)$\}$. *The contribution of any* spin^c *structure associated with* $L + L'$ *to SW invariants is the same as* $SW_X(L)$, *and* $SW_{X\#Y}(L + L') = |H_1(Y, \mathbf{Z}_2)| SW_X(L)$. (2) *The number of* L' *with* $L' \equiv w_2(Y)$ (mod2) *equals* $|H_1(Y, \mathbf{Z})| / |H_1(Y, \mathbf{Z}_2)|$.

These propositions are proved by the standard Uhlenbeck theory. In either case the value of the invariant for $X\#Y$ is the product of that for X and the contributions from flat connections on Y. But to treat the case when $H_1(Y, \mathbf{Z})$ has 2-torsions we need the following observations [11].

(1) For any $w' \in H^2(Y, \mathbf{Z})$ there exists a unique flat $SO(3)$ bundle over Y with $w_2 \equiv w'$ (mod2). Any $SO(3)$-bundle P over $X\#Y$ with $w_2(P) \equiv w + w' \in H^2(X, \mathbf{Z}) \oplus H^2(Y, \mathbf{Z})$ (mod2) is the sum of the $SO(3)$-bundle P_X over X with $w_2(P_X) \equiv w$ (mod2) and the flat $SO(3)$ bundle P_Y with $w_2(P_Y) \equiv w'$ (mod 2).

(2) The moduli spaces of ASD connections over any bundle P over $X\#Y$ in (1) for a generic path of metrics have no $SO(2)$ nor $O(2)$ reducible connections, and hence $D^{w+w'}_{X\#Y}$ is well-defined after the Morgan-Mrowka trick.

(3) $\mathcal{G}_P$ is the kernel of some map from $Aut\ P$ to $H^1(X\#Y, \mathbf{Z}_2)$. In our case we can see by obstruction theory that this map is surjective.

In Donaldson's case we can see that the contribution from the conjugacy classes of the $SO(3)$ representations of $\pi_1 Y$ to the intersection of the space of ASD connections modulo $AutP$ and V_z equals $|H_1(Y, \mathbf{Z})| / |H_1(Y, \mathbf{Z}_2)|$. But $AutP$ acts freely on the space of ASD connections by (2) and $AutP_X = \mathcal{G}_{P_X}$ since $H_1(X, \mathbf{Z}) = 0$, so the contribution from Y to $\mathcal{M}_P \cap V_z$ is $|H_1(Y, \mathbf{Z})|$ by (3). In Seiberg-Witten's case, the contribution of any spin^c structure on Y is 1 because there is no obstruction to constructing the solution from the pair of SW solution for X and that for Y, which is a pair of a flat connection and a zero spinor. We also note that $w_2(Y)$ is a mod 2 reduction of some element in $H^2(Y, \mathbf{Z})$. Thus we obtain the desired result.

Remark. In [10] the contribution of Y (denoted by c_G) to the space of ASD connections modulo the full gauge group, which equals $|H_1(Y, \mathbf{Z})| / |H_1(Y, \mathbf{Z}_2)|$, is considered when $\pi_1 Y = G$ is the fundamental group of a spherical 3-manifold.

§2. Examples of exotic free actions.

First consider a nucleus $N(k)$ for $k \in \mathbf{Z}$ ([4]), whose framed link picture is given by the union of the trefoil knot with framing 0 and its meridian with framing $-k$. Any $N(k)$ contains a regular neighborhood $N(f)$ of a cusp fiber f of the elliptic surfaces, and $N(f)$ contains a 2-torus T of square 0 (a general fiber). For any 4-manifold X containing $N(k)$, denote by X_p (resp. $N(k)_p$) the resulting manifold after p-surgery along T on X (resp. $N(k)$)([3], [9]). In $N(k)_p$ and in X_p there is a multiple fiber f_p such that pf_p is homologous to f. Now we consider a pair of closed oriented 4-manifolds (X, Y) satisfying the following conditions.

(i) $H_1(X, \mathbf{Z}) = 0$, $b_2^+(X) > 1$, $N(k) \subset X$, and X has a SW class.

(ii) Y is a rational homology 4-sphere with an epimorphism from $\pi_1 Y$ to a nontrivial finite group G such that the associated G-covering $\tilde{Y}$ of Y is of the form $S^2 \times S^2 \# Z$ for some 4-manifold Z.

Proposition 2.1 [4]. *$N(k)_p$ is spin if and only if k is even and p is odd. There is a homeomorphism between $N(k)_p$ and $N(k)_{p'}$ inducing the identity on the boundaries if and only if both of them are spin or both of them are non-spin. X_p and $X_{p'}$ are homeomorphic under the same condition.*

Proposition 2.2 [7], [5]. *There is a diffeomorphism between $N(k)_p \# S^2 \times S^2$ and $N(k)_{p'} \# S^2 \times S^2$ inducing the identity on the boundaries and also a diffeomorphism between $X_p \# S^2 \times S^2$ and $X_{p'} \# S^2 \times S^2$, if and only if k, p, p' satisfy the same condition as in* (2-1).

Proposition 2.3 [3], [9]. *The SW classes for X_p are given by* $\{L + (p - 2a - 1)f_p \mid 0 \le a \le p - 1$, *$L$ is a SW class for X*$\}$ *with* $SW_{X_p}(L + (p - 2a - 1)f_p) = SW_X(L)$. *Here* $L \cdot f = L \cdot T$

$= 0$ *and* L *belongs to both* $H^2(X, \mathbf{Z})$ *and* $H^2(X_p, \mathbf{Z})$.

Note that X and X_p are SW simple [9], Corollary 1.6. Next consider the coverings $\widetilde{X_p \# Y}$ of $X_p \# Y$ associated with $\pi_1(X_p \# Y) \to \pi_1(Y) \to G$.

Proposition 2.4. (1) $X_p \# Y$ *and* $X_{p'} \# Y$ *are not diffeomorphic if* $p \neq p'$. (2) $X_p \# Y$ *and* $X_{p'} \# Y$ *are homeomorphic and also* $\widetilde{X_p \# Y}$ *and* $\widetilde{X_{p'} \# Y}$ *are diffeomorphic under the same condition as in Proposition* 2.2.

Proof. (1) comes from (1-2) and (2-3), which show that the numbers of SW classes for $X_p \# Y$'s are different for different p's since f_p is not a torsion class. The first part of (2) comes from (2-1). Finally we have $\widetilde{X_p \# Y} = \tilde{Y} \# |G| X_p = Z \# S^2 \times S^2 \# |G| X_p$ and apply (2-2) on each X_p summand successively to show the rest.

The typical examples satisfying (i) are 1-connected elliptic surfaces $E(k)$ without multiple fibers which contain $N(k)$ (many other examples are now known). To obtain Y satisfying (ii) consider any rational homology 3-sphere M with an epimorhism from $\pi_1 M$ to G and take an untwisted (resp. a twisted) spin $s(M)$ (resp. $s'(M)$) of M which is obtained from $M \times S^1$ by untwisted (resp. twisted) surgery along a curve $* \times S^1$. Then both $s(M)$ and $s'(M)$ are rational homology 4-spheres with $\pi_1 s(M) = \pi_1 s'(M) = \pi_1 M$. Moreover the coverings $\tilde{M}$ of M and $\widetilde{s^{(\prime)}(M)}$ of $s^{(\prime)}(M)$ associated with $\pi_1(s^{(\prime)} M) \cong \pi_1(M) \to G$ satisfy

Proposition 2.5. $\widetilde{s^{(\prime)}(M)}$ *is diffeomorphic to* $s^{(\prime)}(\tilde{M}) \# (|G| - 1) S^2 \times S^2$.

Proof. There is a cobordism W between $\tilde{M} \times S^1$ and $\widetilde{s^{(\prime)}(M)}$ obtained from $\tilde{M} \times S^1 \times [0, 1]$ by attaching $|G|$ 2-handles h_i along $|G|$ parallel circles $*_i \times S^1 \times \{1\}$ on $\tilde{M} \times S^1 \times \{1\}$, whose framings are all untwisted for $\widetilde{s(M)}$, and all twisted for $\widetilde{s'(M)}$. By sliding h_i ($i \geq 2$) along h_1 we can replace them by the 2-handles attached along the trivial circles with untwisted framings. Hence $\widetilde{s^{(\prime)}(M)}$ is obtained from $s^{(\prime)}(\tilde{M})$ (obtained by h_1) by untwisted surgery on $|G| - 1$ trivial circles. This proves (2-5).

On the other hand Cooper-Long proved

Theorem [1]. *Any nontrivial finite group* G *acts freely on a certain rational homology* 3-*sphere* $\tilde{M}$ *(as an orientation-preserving action).*

For such $\tilde{M}$, the orbit space $M = \tilde{M}/G$ is also a rational homology 3-sphere with epimorphism $\pi_1(M) \to G$ associated with the covering $\tilde{M} \to M$ since $H^1(M, \mathbf{Q}) = H^1(\tilde{M}, \mathbf{Q})^G = 0$. Hence by using M we obtain the main theorem from Proposition 2.4. For example, if $Y = s(M)$ and $X = E(k)$ with k odd and $k > 1$ then $X_p \# Y$ are all homeomorphic, mutually non-diffeomorphic, but $\widetilde{X_p \# Y}$ are all diffeomorphic to $s(\tilde{M}) \# (|G| - 1) S^2 \times S^2 \# |G| E(k) \cong s(\tilde{M}) \# (2k|G| - 1)\mathbf{CP}^2 \# (10k|G| - 1)\overline{\mathbf{CP}}^2$.

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