Orbits of triangles obtained by interior division of sides

By Hajime SATO

Department of Mathematics, Gotemba-minami Highschool

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Abstract: Plane triangles are classified by similarity. Let Ω be the set of these equivalence classes of triangles, and $[ABC] \in \Omega$ be the class of triangles which are similar to ΔABC, Putting $x = \angle A$, $y = \angle B$, $z = \angle C$, $[ABC]$ is represented by a point in $\Pi = \{(x, y, z) \mid x + y + z = \pi, x, y, z > 0\}$. By making interior division of sides of ΔABC, we define an orbit in Π, starting from $[ABC]$. It is determined by a differentiable dynamical system, and is the intersection of Π and the surface $\cot x + \cot y + \cot z =$ const.

Key words: Triangles; interior division; convex closed curve; four-vertex theorem.

1. Introduction. We consider here the set $\boldsymbol{T}$ of all triangles on the Euclidean plane. Triangles in $\boldsymbol{T}$ are classified by similarity. In this note, we say that ΔABC is similar to $\Delta A'B'C'$ and write as $\Delta ABC \simeq \Delta A'B'C'$ if $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. It defines an equivalency. Put

(1.1) $[ABC] = \{\Delta A'B'C' \mid \Delta A'B'C' \simeq \Delta ABC\}$

Obviously $[ABC] \cap [A'B'C'] \neq \emptyset$ if and only if $[ABC] = [A'B'C']$. We define

(1.2) $\Omega = (\boldsymbol{T}/\simeq) = \{[ABC] \mid \Delta ABC \in \boldsymbol{T}\}$.

Note that, in general, $[ABC]$, $[BCA]$, and $[CAB]$ are mutually distinct in Ω.

Write $\angle A = x$, $\angle B = y$, $\angle C = z$, then $[ABC]$ is represented as a point in $\boldsymbol{R}^3$. Ω is idenified with the set

(1.3) $\Pi = \{(x, y, z) \mid x + y + z = \pi, x > 0, y > 0, z > 0\}$.

The class of regular triangles is denoted by a point $(\pi/3, \pi/3, \pi/3)$. Points on the boundary of Π denote degenerate triangles. A point in Π corresponding to $[ABC]$ is denoted also by $[ABC]$.

Consider a triangle $\Delta ABC \in [ABC]$. On each side of it, take the point of interior division with the ratio $t : (1 - t)$, where $0 \leqq t \leqq 1$. The point on the side AB is denoted by $A(t)$. Similarly for $B(t)$ and $C(t)$ on BC and CA, respectively. Put

(1.4) $T_0(ABC) = \{[A(t)B(t)C(t)] \mid 0 \leqq t \leqq 1\}$.

$T_0(ABC)$ is represented by a continuous arc in $\Pi \subset \boldsymbol{R}^3$ which connects $[ABC]$ with $[BCA]$. Obviously $T_0(ABC) \cup T_0(BCA) \cup T_0(CAB)$ is a closed curve in Π. Since $B = A(1)$, $C = B(1)$, $A = C(1)$, we may define $[A(1+t)B(1+t)C(1+t)]$, $0 \leqq t \leqq 1$, as $[B(t)C(t)A(t)]$, $0 \leqq t \leqq 1$. Similarly $[A(2+t)B(2+t)C(2+t)]$ may be defined as $[C(t)A(t)B(t)]$. Now for any $t \in \boldsymbol{R}$, let $[t]$ be the greatest integer not exceeding t. Writing $t^* = t - [t]$, $0 \leqq t^* < 1$, we define

(1.5)
$$[A(t)B(t)C(t)] = \begin{cases} [A(t^*)B(t^*)C(t^*)], & \text{if } [t] = 3m + 0 \text{ for some integer } m, \\ [B(t^*)C(t^*)A(t^*)], & \text{if } [t] = 3m + 1 \text{ for some integer } m, \\ [C(t^*)A(t^*)B(t^*)], & \text{if } [t] = 3m + 2 \text{ for some integer } m. \end{cases}$$

For example, if $-1 < t < 0$, then $[t] = -1 = -3 + 2$ and $t^* = 1 - |t|$. Hence $[A(t)B(t)C(t)] = [C(1-|t|)A(1-|t|)B(1-|t|)]$. By (1.5), we define as a continuation of (1.4),

(1.6) $T(ABC) = \{[A(t)B(t)C(t)] \mid t \in \boldsymbol{R}\}$,

which is represented by a closed curve in Π.

There are some investigations on triangles obtained by interior division of sides of ΔABC, e.g. [4]. However, as far as I know, we have almost no knowledge about the set $T(ABC)$, except the case when $t = 1/2$, where $\Delta B(1/2)C(1/2)A(1/2) \simeq \Delta ABC$.

In this note we investigate the set $T(ABC)$. Establishing some lemmas on 2×2 matrices, we will see that $T(ABC)$ is a continuously differentiable curve, and find the system of differential equations which determines the curve. It shows that $T(ABC)$ is a convex curve, represented by the intersection of Π and the surface

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$$\cot x + \cot y + \cot z = \text{const.}$$

2. Some lemmas. Since Ω is identified with the set $\Pi \subset \boldsymbol{R}^3$, we can introduce naturally a topology in Ω.

The proof of the following lemma is easy and may be omitted.

Lemma 2.1. *Triangles ΔABC and $\Delta A(t)B(t)C(t)$ share the center of gravity in common for any t.*

Take a Cartesian coordinate system. We may suppose that the center of gravity of ΔABC is at the origin. Put the coordinates of the vertices to be $A = (a_1, a_2)$, $B = (b_1, b_2)$, then we have $C = (c_1, c_2) = (-a_1 - b_1, -a_2 - b_2)$. Then we get by an easy calculation, for $0 \leqq t \leqq 1$,

$$\begin{aligned} A(t) &= ((1-t)a_1 + tb_1, (1-t)a_2 + tb_2),\\ B(t) &= ((1-t)b_1 + tc_1, (1-t)b_2 + tc_2)\\ &= (-ta_1 + (1-2t)b_1, -ta_2 + (1-2t)b_2),\\ C(t) &= ((1-t)c_1 + ta_1, (1-t)c_2 + ta_2)\\ &= ((2t-1)a_1 + (t-1)b_1, (2t-1)a_2 + (t-1)b_2),\end{aligned}$$

which is written as

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} (1-t)a_1 + tb_1 & (1-t)a_2 + tb_2 \\ -ta_1 + (1-2t)b_1 & -ta_2 + (1-2t)b_2 \end{pmatrix} = \begin{pmatrix} 1-t & t \\ -t & 1-2t \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

since $\Delta A(t)B(t)C(t)$ is determined by $A(t)$ and $B(t)$. For $\Delta A(1+t)B(1+t)C(1+t) = \Delta B(t)C(t)A(t)$, $0 \leqq t \leqq 1$, we have

$$\begin{aligned} A(1+t) = B(t) &= ((1-t)b_1 + tc_1, (1-t)b_2 + tc_2)\\ &= (-ta_1 + (1-2t)b_1, -ta_2 + (1-2t)b_2),\\ B(1+t) = C(t) &= ((1-t)c_1 + ta_1, (1-t)c_2 + ta_2)\\ &= ((2t-1)a_1 + (t-1)b_1, (2t-1)a_2 + (t-1)b_2),\end{aligned}$$

hence

$$\begin{pmatrix} A(1+t) \\ B(1+t) \end{pmatrix} = \begin{pmatrix} -t & 1-2t \\ 2t-1 & t-1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Similarly we have for $\Delta A(2+t)B(2+t)C(2+t) = \Delta C(t)B(t)A(t)$, $0 \leqq t \leqq 1$,

$$\begin{aligned} A(2+t) = C(t) &= ((1-t)c_1 + ta_1, (1-t)c_2 + ta_2)\\ &= ((2t-1)a_1 + (t-1)b_1, (2t-1)a_2 + (t-1)b_2),\\ B(2+t) = A(t) &= ((1-t)a_1 + tb_1, (1-t)a_2 + tb_2),\end{aligned}$$

and

$$\begin{pmatrix} A(2+t) \\ B(2+t) \end{pmatrix} = \begin{pmatrix} 2t-1 & t-1 \\ 1-t & t \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

We define a matrix $M(t)$, $t \in \boldsymbol{R}$, as follows: Let $t^* = t - [t]$, $0 \leqq t^* < 1$, where $[t]$ is the greatest integer not exceeding t,

$$\text{(2.1)} \quad M(t) = \begin{cases} \begin{pmatrix} 1-t^* & t^* \\ -t^* & 1-2t^* \end{pmatrix}, & \text{if } [t] = 3m+0,\\ \begin{pmatrix} -t^* & 1-2t^* \\ 2t^*-1 & t^*-1 \end{pmatrix}, & \text{if } [t] = 3m+1,\\ \begin{pmatrix} 2t^*-1 & t^*-1 \\ 1-t^* & t^* \end{pmatrix}. & \text{if } [t] = 3m+2.\end{cases}$$

By (2.1) we have for $0 \leqq t \leqq 1$,

$$\text{(2.1')} \quad \begin{cases} M(3m-t) \\ \quad = M(3(m-1) + 2 + (1-t)) = M(2 + (1-t)),\\ M(3m-1-t) \\ \quad = M(3(m-1) + 1 + (1-t)) = M(1 + (1-t)),\\ M(3m-2-t) \\ \quad = M(3(m-1) + (1-t)) = M(1-t)).\end{cases}$$

Lemma 2.2. *Suppose that $0 \leqq s, t \leqq 1$.*

(i) *If $s + t \leqq 1$, then*

$$\text{(2.2)} \quad \begin{cases} M(s)M(t) = M(2+s)M(1+t)\\ \quad = M(1+s)M(2+t) = L(s,t)M(k),\\ M(1+s)M(t) = M(s)M(1+t)\\ \quad = M(2+s)M(2+t) = L(s,t)M(1+k),\\ M(1+s)M(1+t) = M(2+s)M(t)\\ \quad = M(s)M(2+t) = L(s,t)M(2+k),\end{cases}$$

where

$$\text{(2.2')} \quad L(s,t) = 1 - 3st, \quad k = \frac{s+t-3st}{1-3st},$$

(ii) *If $1 < s + t \leqq 2$, then*

$$\text{(2.3)} \quad \begin{cases} M(s)M(t) = M(2+s)M(1+t)\\ \quad = M(1+s)M(2+t) = L(s,t)M(1+k),\\ M(1+s)M(t) = M(s)M(1+t)\\ \quad = M(2+s)M(2+t) = L(s,t)M(2+k),\\ M(1+s)M(1+t) = M(2+s)M(t)\\ \quad = M(s)M(2+t) = L(s,t)M(k),\end{cases}$$

where

$$\text{(2.3')} \quad L(s,t) = 1 - 3(1-s)(1-t),$$
$$1 - k = \frac{(1-s) + (1-t) - 3(1-s)(1-t)}{1 - 3(1-s)(1-t)}.$$

Proof. We prove only for $M(s)M(t)$. Other cases are similarily proved. Obviously

$$M(s)M(t) = \begin{pmatrix} 1-s-t & s+t-3st \\ -s-t+3st & 1-2s-2t+3st \end{pmatrix}.$$

(i) Note that $1 - 3st \geqq t + s - 3st = (t+s) - \frac{3}{4}(s+t)^2 + \frac{3}{4}(s-t)^2 > 0$ if $0 < s + t \leqq 1$. From

$$1 - s - t = L(1-k), \; t + s - 3st = Lk,$$
$$1 - 2s - 2t + 3st = L(1-2k),$$

we obtain the result.

(ii) Since $0 \leqq 1 - s, 1 - t \leqq 1$, we have $(1-s) + (1-t) \leqq 1$. From

$1 - s - t = -Lk,\ s + t - 3st = L(1 - 2k),$
$1 - 2s - 2t + 3st = L(k - 1).$
we obtain the result.

For the inverse matrices we have, by easy calculations.

Lemma 2.3. *For any* $0 \leqq t \leqq 1$, *we have*

$$(2.4)\quad \begin{cases} M(t)^{-1} = \dfrac{1}{3t^2 - 3t + 1} M(2 + (1 - t)) \\ \quad = \dfrac{1}{3t^2 - 3t + 1} M(-t), \\ M(1 + t)^{-1} = \dfrac{1}{3t^2 - 3t + 1} M(1 + (1 - t)) \\ \quad = \dfrac{1}{3t^2 - 3t + 1} M(-1 - t), \\ M(2 + t)^{-1} = \dfrac{1}{3t^2 - 3t + 1} M(1 - t) \\ \quad = \dfrac{1}{3t^2 - 3t + 1} M(-2 - t). \end{cases}$$

In connection with (2.2'), we consider the functional equation

$$(2.5)\quad \phi(\sigma + \tau) = \frac{\phi(\sigma) + \phi(\tau) - 3\phi(\sigma)\phi(\tau)}{1 - 3\phi(\sigma)\phi(\tau)}, \quad 0 \leqq \sigma, \tau \leqq 1.$$

Put $\sigma = 0$. Since $3\phi(\tau)^2 - 3\phi(\tau) + 1 \neq 0$, we have that $\phi(0) = 0$. Differentiating with respect to σ and puting $\sigma = 0$, we obtain

$$\frac{dy}{d\tau} = \phi'(0)(3y^2 - 3y + 1),\ y = \phi(\tau).$$

Using $\phi(0) = 0$, requiring that $\phi(1) = 1$, we obtain as a solution of (2.5)

$$(2.6)\quad \phi_0(\sigma) = \frac{2\tan(\frac{2\pi}{3}\sigma)}{\sqrt{3} + 3\tan(\frac{2\pi}{3}\sigma)} = \frac{2\sin(\frac{2\pi}{3}\sigma)}{\sqrt{3}\cos(\frac{2\pi}{3}\sigma) + 3\sin(\frac{2\pi}{3}\sigma)},\ 0 \leqq \sigma \leqq 1,$$

by taking $\phi'_0(0) = 4\pi\ \sqrt{3}/9 = 2.4184\ldots$, $\phi_0(\sigma)$ satisfies (2.5) if $0 \leqq \sigma, \tau, \sigma + \tau \leqq 1$.

For any $\sigma \in \boldsymbol{R}$ we define

$$(2.6')\quad \phi(\sigma) = [\sigma] + \phi_0(\sigma - [\sigma]).$$

It is easy to see that $\phi(\sigma)$ is strictly monotone increasing and satisfies

$$(2.6'')\quad \phi(0) = 0,\ \phi(\tfrac{1}{2}) = \tfrac{1}{2},\ \phi(1) = 1,$$

$$\phi(\tfrac{1}{2} + \sigma) + \phi(\tfrac{1}{2} - \sigma) = 1\ (0 \leqq \sigma \leqq \tfrac{1}{2}).$$

For any $s \in \boldsymbol{R}$ let $s^* = s - [s]$, $s^* = \phi(\sigma^*)$, $0 \leqq \sigma^* < 1$. Putting $\sigma = [s] + \sigma^*$, we get $s = \phi(\sigma)$. Note that the function $\phi(\sigma)$ defined by (2.6') is of the class C^1 but not of C^2.

Put

$$(2.7)\quad K(\sigma) = M(\phi(\sigma)),$$

for $\sigma \in \boldsymbol{R}$. Write

$s = \phi(\sigma) = [s] + s^*,\ t = \phi(\tau) = [t] + t^*,$
$0 \leqq s^*,\ t^* < 1,\ s^* = \phi(\sigma^*),\ t^* = \phi(\tau^*).$

If $s^* + t^* \leqq 1$, then we see that $\sigma^* + \tau^* \leqq 1$ by (2.6"). Using (2.1) and (2.1'), we get from (2.2) and (2.2') that, by (5.5),

$$(2.8)\quad K(\sigma)K(\tau) = L(s^*, t^*)K(\sigma + \tau),$$

where $L(s^*, t^*)$ is the constant in (2.2'). Note that $\phi(\sigma)$ satisfies the functional equation (2.5) *only when* $0 \leqq \sigma, \tau, \sigma + \tau \leqq 1$, *not for general* σ, $\tau \in \boldsymbol{R}$.

Now we consider (2.3) and (2.3'). For $0 \leqq s$, $t \leqq 1$, $s + t > 1$, we put $1 - s = \phi(\sigma')$, $1 - t = \phi(\tau')$ and consider the equation, in connection with (2.3'),

$$(2.5')\quad \phi(\sigma' + \tau') = \frac{\phi(\sigma') + \phi(\tau') - 3\phi(\sigma')\phi(\tau')}{1 - 3\phi(\sigma')\phi(\tau')}, \quad 0 \leqq \sigma', \tau' \leqq 1.$$

As in (2.5) we obtain

$$\phi(\sigma') = \frac{2\tan\left(\frac{2\pi}{3}\sigma'\right)}{\sqrt{3} + 3\tan\left(\frac{2\pi}{3}\sigma'\right)}$$

Put $\sigma = 1 - \sigma'$. Then, using the addition formula of the tangent function,

$$(2.9)\quad 1 - \phi(1 - \sigma) = \frac{\sqrt{3} + \tan\left(\frac{2\pi}{3} - \frac{2\pi}{3}\sigma\right)}{\sqrt{3} + 3\tan\left(\frac{2\pi}{3} - \frac{2\pi}{3}\sigma\right)} = \frac{2\tan\left(\frac{2\pi}{3}\sigma\right)}{\sqrt{3} + 3\tan\left(\frac{2\pi}{3}\sigma\right)} = \phi(\sigma).$$

Since $s + t > 1$, we have $(1 - s) + (1 - t) < 1$. As in (2.6") we see that $\sigma' + \tau' < 1$. Thus $\sigma + \tau = (1 - \sigma') + (1 - \tau') > 1$. By (2.9), for $0 < \sigma + \tau - 1 \leqq 1$,

$1 - \phi(\sigma' + \tau') = 1 - \phi(2 - (\sigma + \tau))$
$= 1 - \phi(1 - (\sigma + \tau - 1)) = \phi(\sigma + \tau - 1).$

By (2.6') we get, if $0 \leqq \sigma, \tau \leqq 1, \sigma + \tau > 1$,

$$\phi(\sigma+\tau)=1+\phi(\sigma+\tau-1)$$
$$=1+(1-\phi(\sigma'+\tau')).$$

By (2.7) we obtain $K(\sigma+\tau)=M(1+\phi(\sigma+\tau-1))$ in this case. Therefore, using (2.1) and (2.1'), we see from (2.3) and (2.3') that (2.8) holds also for the case $s^*+t^*>1$. Noting (2.4) we see that (2.8) holds for any $\sigma,\tau\in\boldsymbol{R}$.

Lemma 2.4. *Let* $M(t)$, $\phi(\sigma)$, $K(\sigma)$ *be defined by* (2.1) — (2.1'), (2.6) — (2.6'), *and* (2.7), *respectively, then we have*

$$K(\sigma)K(\tau)=\text{const.}\ K(\sigma+\tau),\ \sigma,\tau\in\boldsymbol{R}.$$

3. Orbits of triangles obtained by interior division of sides. We have defined triangles $\Delta A(t)B(t)C(t)$, $t\in\boldsymbol{R}$, from the original ΔABC by interior division of sides. Their equivalence classes $[A(t)B(t)C(t)]$ are represented by the matrices $M(t)$ in (2.1). $M(t)$ may be replaced by $K(\tau)=M(\phi(\tau))$ in (2.7).

The class $[A(t)B(t)C(t)]=[A(\phi(\tau))B(\phi(\tau))C(\phi(\tau))]$ is represented by the point $\mathfrak{p}(\tau)=(x(\tau),y(\tau),z(\tau))$ in the set Π in (1.3). We write the point $\mathfrak{p}(\tau)$ as

$\mathfrak{p}(\tau)=\mathfrak{T}(\tau,[ABC])$, denoting that it has originated from ΔABC.

Then $\mathfrak{T}(\sigma,[A(t)B(t)C(t)])$ denotes the class of triangles obtained from $\Delta A(t)B(t)C(t)$, in place of ΔABC, by interior division of sides with the ratio $s:(1-s)$, $s=\phi(\sigma)\in[0,1)$. Since a const. multiplication does not alter the similarity of triangles, we obtain, by Lemma 2.4:

Theorem 3.1. *With the notations stated above, we have*

$$\mathfrak{T}(\sigma,\mathfrak{T}(\tau,[ABC]))=\mathfrak{T}(\sigma+\tau,[ABC]),\ \sigma,\tau\in\boldsymbol{R},\ \mathfrak{T}(0,[ABC])=[ABC].$$

Thus $\mathfrak{T}(\tau,[ABC])$ *forms a* 1*-parameter group, hence defines a dynamical system.*

The set $T(ABC)$ *in* (1.6) *is the trajectory of* $\mathfrak{T}(\tau,[ABC])$, *which is a simple closed curve. If* $T(ABC)\cap T(A'B'C')\neq\emptyset$, *then* $T(ABC)=T(A'B'C')$.

The interior of the set Π in (1.3) is filled up with these closed trajectories $T(ABC)$.

Suppose $0\leqq t\leqq 1$. The angles $\angle A(t)$ and $\angle A(1+t)$ at the vertices $A(t)$, $A(1+t)$ of $\Delta A(t)B(t)C(t)$ and $\Delta A(1+t)B(1+t)C(1+t)$, respectively, are given by

$$\text{(3.1)}\quad \cos(\angle A(t))=\frac{f(t)}{g(t)h(t)},$$
$$\cos(\angle A(1+t))=\frac{f(1+t)}{g(1+t)h(1+t)},$$

in which we put, writing $p=a_1^2+a_2^2$, $q=a_1b_1+a_2b_2$, $r=b_1^2+b_2^2$,

$$f(t)=\overrightarrow{A(t)B(t)}\cdot\overrightarrow{A(t)C(t)}$$
$$=(2-3t)p-(9t^2-9t+1)q+(3t-1)r,$$
$$g(t)=|\overrightarrow{A(t)B(t)}|=\sqrt{p+2(3t-1)q+(3t-1)^2r},$$
$$h(t)=|\overrightarrow{A(t)C(t)}|=\sqrt{(3t-2)^2p-2(3t-2)q+r},$$

and

$$f(1+t)=\overrightarrow{A(1+t)B(1+t)}\cdot\overrightarrow{A(1+t)C(1+t)}$$
$$=(3t-1)p+(9t^2-3t-1)q+(9t^2-9t+2)r,$$
$$g(1+t)=|\overrightarrow{A(1+t)B(1+t)}|$$
$$=\sqrt{(3t-1)^2p+2(9t^2-9t+2)q+(3t-2)^2r},$$
$$h(1+t)=|\overrightarrow{A(1+t)C(1+t)}|$$
$$=\sqrt{p+(6t-2)q+(9t^2-6t+1)r}.$$

Then

$$\lim_{t\to1-0}\frac{f'(t)}{f(t)}-\lim_{t\to+0}\frac{f'(1+t)}{f(1+t)}=6,$$
$$\lim_{t\to1-0}\frac{g'(t)}{g(t)}-\lim_{t\to+0}\frac{g'(1+t)}{g(1+t)}=3,$$
$$\lim_{t\to1-0}\frac{h'(t)}{h(t)}-\lim_{t\to+0}\frac{h'(1+t)}{h(1+t)}=3.$$

Since

$$\frac{d}{dt}\log\cos(\angle A(t))=\frac{f'(t)}{f(t)}-\frac{g'(t)}{g(t)}-\frac{h'(t)}{h(t)},$$
$$\frac{d}{dt}\log\cos(\angle A(1+t))$$
$$=\frac{f'(1+t)}{f(1+t)}-\frac{g'(1+t)}{g(1+t)}-\frac{h'(1+t)}{h(1+t)},$$

and $\lim_{t\to1-0}\cos(\angle A(t))=\lim_{t\to+0}\cos(\angle A(1+t))$, we obtain that

$$\lim_{t\to1-0}\frac{d}{dt}\cos(\angle A(t))=\lim_{t\to+0}\frac{d}{dt}\cos(\angle A(1+t)).$$

Similarly

$$\lim_{t\to1-0}\frac{d}{dt}\cos(\angle A(1+t))=\lim_{t\to+0}\frac{d}{dt}\cos(\angle A(2+t)),$$
$$\lim_{t\to1-0}\frac{d}{dt}\cos(\angle A(2+t))=\lim_{t\to+0}\frac{d}{dt}\cos(\angle A(t)).$$

Thus we know that $\angle A(t)$, $t\in\boldsymbol{R}$, is continuously differentiable. Note that the side length $|\overrightarrow{A(t)B(t)}|$ is not so at $t=0,\pm1,\pm2,\ldots$. Similarly for $\angle B(t)$ and $\angle C(t)$.

Since the equivalence class $[A(t)B(t)C(t)]=\mathfrak{T}(\tau,[ABC])$, $t=\phi(\tau)$, is represented by the point $(x(\tau),y(\tau),z(\tau))=(\angle A(t),\angle B(t),\angle C(t))$, we obtain the following theorem:

Theorem 3.2. *The dynamical system* $\mathfrak{T}(\tau,[ABC])$ *is continuously differentiable.*

Now we will obtain the system of differential equations for $\mathfrak{T}(\tau, [ABC])$. Since

$$\frac{d}{d\sigma}\mathfrak{T}(\tau+\sigma, [ABC]) = \frac{d}{d\tau}\mathfrak{T}(\tau+\sigma, [ABC]) = \frac{d}{d\sigma}\mathfrak{T}(\sigma, \mathfrak{T}(\tau, [ABC])),$$

we get, by taking $\sigma = 0$,

$$\text{(3.2)}\quad \frac{d}{d\tau}\mathfrak{T}(\tau, [ABC]) = \frac{d}{d\sigma}\mathfrak{T}(\sigma, \mathfrak{T}(\tau, [ABC]))\,|_{\sigma=0}.$$

We will find $\frac{d}{d\sigma}\mathfrak{T}(\sigma, [ABC])\,|_{\sigma=0}$. Using the same notations as above, we get by (3.1)

$$\text{(3.2')}\quad \frac{d}{dt}\cos(\angle A(t))\,|_{t=0} = \frac{27(pr-q^2)(p+2p)}{(p-2q+r)^{3/2}(4p+4q+r)^{3/2}}.$$

By an elementary calculation, writing $a = |\overrightarrow{BC}|$, $b = |\overrightarrow{CA}|$, $c = |\overrightarrow{AB}|$, $a^2 = p + 4q + 4r$, $b^2 = 4p + 4q + r$, $c^2 = p - 2q + r$, hence

$$p = \frac{1}{9}(-a^2 + 2b^2 + 2c^2),$$

$$q = \frac{1}{18}(a^2 + b^2 - 5c^2),\ r = \frac{1}{9}(2a^2 - b^2 + 2c^2).$$

Write $\alpha = \sin(\angle A)$, $\beta = \sin(\angle B)$, $\gamma = \sin(\angle C)$. Using the sine theorem: $a/\alpha = b/\beta = c/\gamma$, we have by (3.2')

$$\text{(3.3)}\quad \alpha\frac{d}{dt}\angle A(t)\,|_{t=0} = \frac{(\beta^2-\gamma^2)(\alpha^4+\beta^4+\gamma^4-2\beta^2\gamma^2-2\gamma^2\alpha^2-2\alpha^2\beta^2)}{4\beta^3\gamma^3}.$$

Hence by (3.2) and (3.3), substituting $\Delta A(t)B(t)C(t)$ for ΔABC, we obtain the required system of differential equations. Writing $u = \sin x(\tau)$, $v = \sin y(\tau)$, $w = \sin z(\tau)$, where $(x(\tau), y(\tau), z(\tau)) = (\angle A(t), \angle B(t), \angle C(t))$, $t = \phi(\tau)$, we get, with $\phi'(0) = (4\pi\sqrt{3})/9$,

$$\text{(3.4)}\quad \begin{cases} \dfrac{dx}{d\tau} = \phi'(0)\dfrac{(v^2-w^2)(u^4+v^4+w^4-2v^2w^2-2w^2u^2-2u^2v^2)}{4uv^3w^3} = u^2(v^2-w^2)S, \\[2ex] \dfrac{dy}{d\tau} = \phi'(0)\dfrac{(w^2-u^2)(u^4+v^4+w^4-2v^2w^2-2w^2u^2-2u^2v^2)}{4vw^3u^3} = v^2(w^2-u^2)S, \\[2ex] \dfrac{dz}{d\tau} = \phi'(0)\dfrac{(u^2-v^2)(u^4+v^4+w^4-2v^2w^2-2w^2u^2-2u^2v^2)}{4wu^3v^3} = w^2(u^2-v^2)S, \end{cases}$$

where $u = \sin x$, $v = \sin y$, $w = \sin z$ and

$$\text{(3.4')}\quad S = S(u, v, w) = \frac{4\pi}{3\sqrt{3}}\,\frac{u^4+v^4+w^4-2v^2w^2-2w^2u^2-2u^2v^2}{4u^3v^3w^3}.$$

Since $uvw \neq 0$ in Π, and a trajectory of (3.4) remains in Π if it starts at a point of the same set, we know by [2, p. 34, Theorem 8.1] that

(3.5) $\mathfrak{T}(\tau, [ABC])$ is analytic with respect to τ.

We note that $t = \phi(\tau)$ is a C^1 (but not C^2) function of τ.

The only fixed point of (3.4) in Π is $u = v = w$, that is $x = y = z = \pi/3$. Further in (3.4), if we change v and w (hence y and z) and τ to $-\tau$, we obtain the same system. Hence the trajectory is symmetric with respect to the plane $y - z = 0$. Similarly for $z - x = 0$, $x - y = 0$. Therefore we have, identifying $\mathfrak{T}(\tau, [ABC])$ with its trajectory $T(ABC)$.

Theorem 3.3. *Trajectory $\mathfrak{T}(\tau, [ABC])$ is a closed curve which is symmetric with respect to the planes $y - z = 0$, $z - x = 0$, $x - y = 0$. Therefore it encircles the point $(\pi/3, \pi/3, \pi/3)$.*

It degenerates to one point if and only if ΔABC is a regular triangle.

Note that, for the system (3.4), the eigenvalues at the fixed point $(\pi/3, \pi/3, \pi/3)$ are 0 (with eigenvector orthogonal to Π) and pure imaginary numbers.

Now we will show the trajectories are convex. Put $\mathfrak{p}(\tau) = \mathfrak{T}(\tau, [ABC])$. Since $\mathfrak{p}(\tau)$ is a plane curve on Π, its curvature $\kappa(\tau)$ is given by

$$\text{(3.6)}\quad \kappa(\tau) = \text{the component of } \frac{\dot{\mathfrak{p}}(\tau)\times\ddot{\mathfrak{p}}(\tau)}{|\dot{\mathfrak{p}}(\tau)|^3},$$

orthogonal to Π,

where $\dot{}$ denotes differentiation with respect to τ. If we would prove that $\kappa(\tau) > 0$, then the trajectory $\mathfrak{p}(\tau)$ should be known to be convex. [1]

Since $\mathfrak{h} = {}^t(1, 1, 1)$ is orthogonal to Π, we aim to show that

$$(\dot{\mathfrak{p}}(\tau)\times\ddot{\mathfrak{p}}(\tau))\cdot\mathfrak{h} = \begin{vmatrix}\dot y & \dot z\\ \ddot y & \ddot z\end{vmatrix} + \begin{vmatrix}\dot z & \dot x\\ \ddot z & \ddot x\end{vmatrix} + \begin{vmatrix}\dot x & \dot y\\ \ddot x & \ddot y\end{vmatrix} > 0.$$

Consider the system (3.4). Put, with $u = \sin x$, v

$= \sin y$ and $w = \sin z$,

$$X = u^2(v^2 - w^2),\ Y = v^2(w^2 - u^2),$$
$$Z = w^2(u^2 - v^2).$$

Then, with $S = S(u, v, w)$ in (3.4'),

$$\begin{vmatrix} \dot{y} & \dot{z} \\ \ddot{y} & \ddot{z} \end{vmatrix} = \begin{vmatrix} YS & ZS \\ \dot{Y}S + Y\dot{S} & \dot{Z}S + Z\dot{S} \end{vmatrix}$$
$$= \begin{vmatrix} Y & Z \\ \dot{Y} & \dot{Z} \end{vmatrix} (S(u, v, w))^2,$$

hence it suffices to see positiveness of

$$K = \left(\begin{vmatrix} Y & Z \\ \dot{Y} & \dot{Z} \end{vmatrix} + \begin{vmatrix} Z & X \\ \dot{Z} & \dot{X} \end{vmatrix} + \begin{vmatrix} X & Y \\ \dot{X} & \dot{Y} \end{vmatrix} \right) (S(u, v, w))^2.$$

By an easy calculation we see

$$\begin{vmatrix} Y & Z \\ \dot{Y} & \dot{Z} \end{vmatrix} + \begin{vmatrix} Z & X \\ \dot{Z} & \dot{X} \end{vmatrix} + \begin{vmatrix} X & Y \\ \dot{X} & \dot{Y} \end{vmatrix}$$
$$= 2(\dot{u}(w^2 - v^2)uv^2w^2 + \dot{v}(u^2 - w^2)u^2vw^2 + \dot{w}(v^2 - u^2)u^2v^2w).$$

Since $\dot{u} = \cos x\, \dot{x}$ etc., we obtain by (3.4)

(3.7) $K = -6u^2v^2w^2F(x, y, z)(S(u, v, w))^3$,

where

$F(x, y, z) = \cos x(v^2 - w^2)^2u + \cos y(w^2 - u^2)^2v + \cos z(u^2 - v^2)^2w$

$= \cos x \sin x\, (\sin^2 y - \sin^2 z)^2 + \cos y \sin y\, (\sin^2 z - \sin^2 x)^2 + \cos z \sin z(\sin^2 x - \sin^2 y)^2$.

First we note that $u, v, w > 0$ and $u \pm v \pm w \neq 0$ for $(x, y, z) \in \Pi$. Hence

$$u^4 + v^4 + w^4 - 2v^2w^2 - 2w^2u^2 - 2u^2v^2$$
$$= (u + v + w)(u + v - w)(u - v + w)(u - v - w) \neq 0$$

and $S(u, v, w) \neq 0$ for $(x, y, z) \in \Pi$. On the other hand $S(u, v, w) < 0$ for $x = y = z = \pi/3$, therefore $S(u, v, w) < 0$ in Π.

Next note that $\min\{F(x, y, z) \mid x, y, z \geqq 0, x + y + z = \pi\}$ is attained at $P_0 = (\pi/3, \pi/3, \pi/3)$ and equals 0. Thus $F(x, y, z) > 0$ for $(x, y, z) \in \Pi \setminus \{P_0\}$, which shows that $\kappa(\tau) > 0$. Hence

Theorem 3.4. *Trajectory $T(ABC)$ is a convex closed curve if it is not one point.*

By the well known Four-Vertex Theorem [1], $\mathfrak{T}(\tau, [ABC])$ possesses at least 4 vertices. In fact, it admits 6 vertices at intersections with the planes $x - y = 0$, $y - z = 0$, $z - x = 0$.

By (3.4) we see

$$\frac{dx}{d\tau} + \frac{dy}{d\tau} + \frac{dz}{d\tau} = 0,\ \frac{1}{u^2}\frac{dx}{d\tau} + \frac{1}{v^2}\frac{dy}{d\tau} + \frac{1}{w^2}\frac{dz}{d\tau} = 0.$$

Since $u = \sin x$, $v = \sin y$, $w = \sin z$, we obain that

(3.8) $x + y + z = \text{const.} = \pi$,

(3.9) $\cot x + \cot y + \cot z = \text{const.}$

Therefore we obtain the following Theorem :

Theorem 3.5. *Trajectory $T(ABC)$ is given by the intersection of surfaces* (3.8) *and* (3.9), *where the* const. *in* (3.9) *is equal to* $\cot(\angle A) + \cot(\angle B) + \cot(\angle C)$.

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