Interacting Brownian motions with measurable potentials

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1. Introduction. We construct (unlabeled) interacting Brownian motions, so-called infinite dimensional Wiener processes with interaction, by using Dirichlet form theory.

(Labeled) interacting Brownian motions are infinitely dimensional diffusion processes with state space $(\boldsymbol{R}^d)^N$ given by the following SDE;

$$(1.1)\quad dX_t^i = dB_t^i - \sum_{j=1, j\neq i}^{\infty} \frac{1}{2}\nabla\Phi(X_t^i - X_t^j)dt \qquad (1 \le i < \infty),$$

where B_t^i are independent Brownian motion on $\boldsymbol{R}^d$. When $\Phi \in C_0^3(\boldsymbol{R}^d)$, this equation was solved by Lang [3], [4]. (see [1], [5], [8], [11], [12] for further development). The Θ-valued diffusion process associated with (1.1) (unlabeled interacting Brownian motion) is

$$\Xi_t = \sum_{i=1}^{\infty}\delta_{X_t^i} \ (\delta_a \text{ is the delta measure at } a).$$

Diffusion processes $\{P_\theta\}_{\theta\in\Theta}$ obtained in Corollary 1.3 below is corresponds to Ξ_t. We refer to Theorem 3 in [7] with Remark (3,4) after that for the precise meaning of *correspondence* and related open problems.

We assume interacting potential Φ is super stable and lower regular in the sense of Ruelle, and there exists a upper semicontinuous function $\tilde{\Phi}$ that are regular in the sense of Ruelle and dominates Φ from above. We remark Φ itself is not necessarily upper semicontinuous; Φ needs no regularity more than measurability. We henceforth generalize results in [7] and [13].

Let Θ be the set of all locally finite configurations on $\boldsymbol{R}^d$. Here a configuration θ is a Radon measure of the form $\theta = \sum_i \delta_{x^i}$, where $\{x^i\}$ is a finite or infinite sequence in $\boldsymbol{R}^d$ with no cluster points. We endow Θ with the vague topology; Θ is a Polish space with this topology.

Let Φ: $\boldsymbol{R}^d \to \boldsymbol{R} \cup \{\infty\}$ be a measurable function such that $\Phi(x) = \Phi(-x)$. We assume:

(Φ.1) Φ is super stable in the sense of Ruelle. (see [9] and [10]).

(Φ.2) Φ is lower regular in the sense of Ruelle; there exist a positive, decreasing function φ: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ satisfying

$$\int_{\boldsymbol{R}^+}\varphi(t)t^{d-1}dt < \infty,$$
$$\Phi(x) \ge -\varphi(|x|) \text{ for all } x \in \boldsymbol{R}^d.$$

(Φ.3) There exists a upper semicontinuous function $\tilde{\Phi}$: $\boldsymbol{R}^d \to \boldsymbol{R} \cup \{\infty\}$ and a constant $R > 0$ such that

$$\Phi(x) \le \tilde{\Phi}(x) \text{ for all } x \in \boldsymbol{R}^d,$$
$$\tilde{\Phi}(x) \le \varphi(|x|) \text{ for all } |x| \ge R,$$
$$\tilde{\Phi}(x) = \infty \text{ if and only if } \Phi(x) = \infty.$$

Here φ is same as (Φ.2).

We remark by (Φ.1) Φ is bounded from below. By (Φ.1) $-$ (Φ.3) for each $z > 0$ there exist (grand canonical) Gibbs measures μ with pair potential Φ and activity z ([10]). The definition of Gibbs measure will be given in Section 2.

We consider a symmetric bilinear form $\mathscr{E}$ on Θ;

$$\mathscr{E}(f, g) = \int_\Theta D[f, g]d\mu.$$

Here $D[f, g]$ is given by

$$D[f, g](\theta) = \frac{1}{2}\sum_i \nabla_i\hat{f}(x)\cdot\nabla_i\hat{g}(x).$$

Here $\nabla_i = (\frac{\partial}{\partial_{x_{ik}}})_{1\le k\le d}$, and $\cdot$ means the inner product on $\boldsymbol{R}^d$. $\hat{f}$ and $\hat{g}$ in the right hand side are permutation invariant functions given by $f(\theta) = \hat{f}(x)$ and $g(\theta) = \hat{g}(x)$, where $x = (x^i)$ is such that $\theta = \sum_i\delta_{x^i}$. Bilinear map $D[f, g]$ is defined on $\mathscr{D}_\infty^{loc.}$, the space of local, smooth functions on Θ, defined in Section 2. Let

$$\mathscr{D} = \{f \in \mathscr{D}_\infty^{loc.}; \mathscr{E}(f, f) < \infty, \|f\|_{L^2(\Theta,\mu)} < \infty\}.$$

The purpose of this paper is to prove $(\mathscr{E}, \mathscr{D})$ is closable on $L^2(\Theta, \mu)$.

Theorem. 1.1. *Assume* (Φ.1) $-$ (Φ.3). *Let μ be a Gibbs measure with potential Φ. Then $(\mathscr{E}, \mathscr{D}_\infty)$ is closable on $L^2(\Theta, \mu)$.*

Remark 1.1. In the previous work [7] we proved this result under more restrictive assumptions (Φ.1), (Φ.2) and (Φ.3′), (Φ.4′) below:

(Φ.3′) Φ is tempered in the sense of Ruelle; there exist a decreasing function φ: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ and a constant R_1 such that

$$\int_{\mathbf{R}^+} \varphi(t) t^{d-1} dt < \infty,$$

$$\Phi(x) \le \varphi(|x|) \text{ for all } |x| \ge R_1.$$

(Φ.4′) Φ is upper semicontinuous on $\{x; \Phi(x) < \infty\}$, and $\{x; \Phi(x) = \infty\} = \phi$ or $\{x; \Phi(x) = \infty\} = \{|x| \le R_2\}$ for some R_2.

We see (Φ.3′) and (Φ.4′) are stronger than (Φ.3). Indeed, we can construct $\hat{\Phi}$ as follows; $\hat{\Phi}(x) = \max\{\Phi(x), \hat{\varphi}(|x|)\}$. Here $\hat{\varphi}$ is the left continuous version of φ.

Remark 1.2. For the existence of Gibbs measures (Φ.1), (Φ.2), and (Φ.3″),

(Φ.3″) $\int_{\mathbf{R}^d} |1 - e^{-\Phi(x)}| dx < \infty.$

are enough. It, however, seems difficult to replace (Φ.3) by (Φ.3″). For even if the number of particle is finite, we need a certain assumption on density of reference measures.

Let $(\mathscr{E}, \mathscr{D})$ denote the closure obtained by Theorem 1.1. Let $\Theta_r^i = \{\theta \in \Theta; \theta(\mathscr{Q}_r) = i\}$, where $\mathscr{Q}_r = \{|x| \le r\}$. Let σ_r^i denote density functions of μ given by (2.2). In order to prove the quasi regularity of $(\mathscr{E}, \mathscr{D})$ we need

(μ.1) $\sum_{i=1}^{\infty} i\mu(\Theta_r^i) < \infty$ for all $r \in \mathbf{N}$,

(μ.2) σ_r^i are bounded for all $i, r \in \mathbf{N}$.

Combining Theorem 1.1 with Theorem 1 in [7] we obtain:

Corollary 1.2. *In addition to the assumptions in Theorem* 1.1, *assume* (μ.1) *and* (μ.2). *Then* $(\mathscr{E}, \mathscr{D})$ *is a quasi regular Dirichlet form on* $L^2(\Theta, \mu)$.

Combining this with results in [6] or ch. 7 in [2] we obtain:

Corollary 1.3. *Assume the same assumptions in Theorem* 1.2. *Then there exists a diffusion* $\{P_\theta\}_{\theta\in\Theta}$ *associated with* $(\mathscr{E}, \mathscr{D})$ *on* $L^2(\Theta, \mu)$. *Moreover* $\{P_\theta\}_{\theta\in\Theta}$ *is reversible with invariant measure* μ.

2. Proof of Theorem 1.1.

A function f on Θ is called local if $\hat{f}$ is $\sigma[\pi_r]$-measurable for some $r \in \mathbf{N}$. Here $\pi_r: \Theta \to \Theta$ is such that $\pi_r(\theta) = \theta(\cdot \cap \{|x| \le r\})$. We say f is smooth if $\hat{f}$ is smooth, where $\hat{f}$ is a permutation invariant function such that $f(\theta) = f(x)$. ($x = (x^i)$ is such that $\theta = \sum_i \delta_{x^i}$). Let $\mathscr{D}_\infty^{loc}$ denote the set of all local, smooth functions on Θ.

Let $\mathscr{Q}_r^i$ denote i-product of $\mathscr{Q}_r = \{|x| \le r\}$. For $x = (x^k) \in \mathscr{Q}_r^i$, $\theta \in \Theta$, $r \le s$, let

$$H_{r,\theta,s}^i(x) = \sum_{1\le k<\ell\le i} \Phi(x^k - x^\ell) + \sum_{\ell\in L(r,s)} \{ \sum_{1\le k\le i} \Phi(x^k - y^\ell)\},$$

where $(y^\ell)_\ell$ is such that $\theta = \sum_\ell \delta_{y^\ell}$ and $L(r, s) = \{\ell; r < |y^\ell| \le s\}$. We set

(2.1) $H_{r,\theta}^i(x) = \lim_{s\to\infty} H_{r,\theta,s}^i(x)$, whenever the limit exists.

We remark $H_{r,\theta}^i$ may be infinite. Let $\mathscr{Q}(z) = \mathscr{Q}_1 + z$, where $z \in \mathbf{Z}^d$. Let

$$\Theta_0 = \{\theta \in \Theta; \sup_{r\in\mathbf{N}} r^{-d} \sum_{z\in Z^d, |z|\le r} \theta(\mathscr{Q}(z))^2 < \infty\}.$$

Recall that Φ is bounded from below and $\Phi(x) \ge -\varphi(|x|)$. So it is easy to see that the limit ($\le \infty$) in (2.1) exists for $\theta \in \Theta_0$. For $z > 0$ let

$$m_{r,\theta}^i(x) = Z_{r,\theta}^{-1} \frac{z^i}{i!} \exp[-H_{r,\theta}^i(x)],$$

$$Z_{r,\theta} = e^{-|\mathscr{Q}_r|} \sum_{i=0}^{\infty} \frac{z^i}{i!} \int_{\mathscr{Q}_r^i} \exp[-H_{r,\theta}^i(x)] dx.$$

Here we set the summand to be 1 for $i = 0$. Let $M_{r,\theta}$ be the $\sigma[\pi_r]$-measurable function defined by

$$M_{r,\theta}(\theta_1) = m_{r,\theta}^i(x) \quad \text{for } \theta_1 \in \Theta_r^i.$$

Here $x = (x^i) \in \mathscr{Q}_r^i$ is such that $\theta_1 = \sum_i \delta_{x^i}$.

Let $\pi_r^c: \Theta \to \Theta$ such that $\pi_r^c(\theta) = \theta \cap (\mathbf{R}^d - \mathscr{Q}_r)$ and let Λ_r denote the Poison random measure with intensity $1_{\mathscr{Q}_r} dx$.

Definition 2.1. A probability measure μ on $(\Theta, \mathscr{B}(\Theta))$ is called a (grand canonical) Gibbs measure with potential Φ and activity z if μ satisfies the following:

(G.1) $\mu(\Theta_0) = 1$ (tempered),

(G.2) $\mu(A|\sigma[\pi_r^c])(\theta) = \int_A M_{r,\theta}(\theta_1) d\Lambda_r(\theta_1)$ for $A \in \sigma[\pi_r]$.

For μ we define density functions as follows.

$$\sigma_r^i(x) = i! \int_\Theta m_{r,\theta}^i(x) d\mu. \tag{2.2}$$

For $f: \Theta \to \mathbf{R}$ let $f_{r,\theta}^i(x): \Theta \times \mathscr{Q}_r^i \to \mathbf{R}$ denote the function satisfying the following;

(1) $f_{r,\theta}^i(x)$ is a permutation invariant function on $\mathscr{Q}_r^i$ for each $\theta \in \Theta$.

(2) $f_{r,\theta(1)}^i(x) = f_{r,\theta(2)}^i(x)$ if $\pi_r^c(\theta(1)) = \pi_r^c(\theta(2))$, $\theta(1), \theta(2) \in \Theta_r^i$.

(3) $f_{r,\theta}^i(x) = f(\theta)$ for $\theta \in \Theta_r^i$, ($x = (x^i)$ is such that $\pi_r(\theta) = \sum_i \delta_{x^i}$).

(4) $f_{r,\theta}^i(x) = 0$ for $\theta \notin \Theta_r^i$.

For a.e. θ, let $\varepsilon_{r,\theta}^i$ denote the bilinear form on $C_b^\infty(\mathscr{Q}_r^i)$ given by

$$\varepsilon_{r,\theta}^i(f, g) = \int_{\mathscr{Q}_r^i} D[f, g](x) m_{r,\theta}^i(x) dx.$$

Lemma 2.1. *For a.e.* θ $(\varepsilon_{r,\theta}^i, C_b^\infty(\mathscr{Q}_r^i))$ *is closable on* $L^2(\mathscr{Q}_r^i, m_{r,\theta}^i dx)$.

Proof. Let $\Gamma_n = \{x \in \boldsymbol{R}^d ; \tilde{\Phi}(x) \geq n\}$ and $\Gamma_n(y) = \Gamma_n + y$. Let $\theta \in \Theta_0$ and $i, r \in \boldsymbol{N}$ be fixed. We set

$$O_n = \{x = (x^k) \in \mathcal{Q}_r^i ;\ \text{dist.}(x^k, \Gamma_n(y^j)) > 0 \text{ for all } k, j,\ \text{dist.}(x^k, \Gamma_n(x^\ell)) > 0 \text{ for all } k \neq \ell\}.$$

Here $\{y^j\}$ is such that $\pi_r^c(\theta) = \sum_j \delta_{y^j}$. Then since $\tilde{\Phi}$ is bounded on $\boldsymbol{R}^d \setminus \Gamma_n$, $|\tilde{\Phi}(x)| \leq \varphi(|x|)$ for $|x| \geq R$ and $\theta \in \Theta_0$, we see

$$\sum_j \sup\{\sum_{k=1}^{i} |\tilde{\Phi}(x^k - y^j)| ; x = (x^k) \in O_n\} < \infty.$$

Hence by Lebesgue's convergence theorem $\sum_j \{\sum_{k=1}^i \tilde{\Phi}(x^k - y^j)\}$ is upper semicontinuous on O_n. So

$$h(x) := \sum_{k,\ell=1}^{i} \tilde{\Phi}(x^k - x^\ell) + \sum_j \{\sum_{k=1}^{i} \tilde{\Phi}(x^k - y^j)\}$$

is also upper semicontinuous on O_n. Since Φ (henceforth $\tilde{\Phi}$) is bounded from below and $\Phi(x) \geq -\varphi(|x|)$, there exists a constant C_1 independent of n such that $h(x) \geq C_1$. Let $H_n(x) = e^{-h(x)}$ if $x \in O_n$, and $H_n(x) = 0$ otherwise. Note that H_n is lower semicontinuous and bounded. Hence the bilinear form $(\tilde{\varepsilon}_n, C_b^\infty(\mathcal{Q}_r^i))$ given by

$$\tilde{\varepsilon}_n(f, g) = \int_{\mathcal{Q}_r^i} D[f, g] H_n(x) dx.$$

is closable on $L^2(\mathcal{Q}_r^i, H_n(x)dx)$ (see Lemma 3.2 in [7] for proof). This implies $(\tilde{\varepsilon}_n, C_b^\infty(\mathcal{Q}_r^i))$ is closable on $L^2(\mathcal{Q}_r^i, m_{r,\theta}^i dx)$ because

(2.3) $H_n \leq m_{r,\theta}^i \leq C_2 H_n$ on O_n for some constant C_2.

Here we used $\Phi \leq \tilde{\Phi}$ for the first inequality, and $\sup_{O_n} m_{r,\theta}^i < \infty$ and $\inf_{O_n} H_n > 0$ for the second.

Let

$$\varepsilon_n(f, g) = \int_{O_n} D[f, g] m_{r,\theta}^i(x) dx.$$

Then by (2.3)

$$\tilde{\varepsilon}_n(f, f) \leq \varepsilon_n(f, f) \leq C_2 \tilde{\varepsilon}_n(f, f).$$

So $(\varepsilon_n, C_b^\infty(\mathcal{Q}_r^i))$ is also closable on $L^2(\mathcal{Q}_r^i, m_{r,\theta}^i dx)$. Moreover, the increasing limit of $\{(\varepsilon_n, C_b^\infty(\mathcal{Q}_r^i))\}_{n \in \mathrm{N}}$ is $(\varepsilon_{r,\theta}^i, C_b^\infty(\mathcal{Q}_r^i))$. Hence $(\varepsilon_{r,\theta}^i, C_b^\infty(\mathcal{Q}_r^i))$ is also closable on $L^2(\mathcal{Q}_r^i, m_{r,\theta}^i dx)$. □

Proof of Theorem 1.1. Lemma 2.1 corresponds to Proposition 4.1 in [7]. So the rest of the proof is exactly same as the proof of Theorem 4 in [7]. □

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