Maximal Unramified Extensions of Imaginary Quadratic Number Fields of Small Conductors

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Let K be an algebraic number field (of finite degree) and K_{ur} its maximal unramified extension. Then the Galois group $\mathrm{Gal}(K_{ur}/K)$ can be both finite and infinite and in general it is quite difficult to determine the structure of this group. If K has sufficiently small root discriminant, then $K_{ur} = K$, that is , K has no nontrivial unramified extension. This is the case, for example, for the imaginary quadratic number fields with class number one, the cyclotomic number fields with class number one, and the real abelian number fields of prime power conductors $\leqq 67$ (see [20, Appendix]). For some fields K with small root discriminant, we can determine $\mathrm{Gal}(K_{ur}/K)$. The purpose of this article is to report that we have determined the structure of $\mathrm{Gal}(K_{ur}/K)$ of imaginary quadratic number fields K of small conductors. (Details will apear elsewhere [21]). For imaginary quadratic number fields K of conductors $\leqq 420$ ($\leqq 719$ under the Generalized Riemann Hypothesis (GRH)) we determine $\mathrm{Gal}(K_{ur}/K)$ and tabulate them for K with $K_{ur} \neq K_1$, where K_1 denotes the Hilbert class field of K. (If $K_{ur} = K_1$, then $\mathrm{Gal}(K_{ur}/K) = \mathrm{Gal}(K_1/K) \cong \mathrm{Cl}(K)$, the class group of K by class field theory). For all such K, $K_{ur} = K$, K_1, K_2, or K_3, where K_2 (resp. K_3) is the second (resp. third) Hilbert class field of K. In other words, K_{ur} coincides with the top of the class field tower of K and the length of the tower is at most three. If possible, we give also simple expressions of K_1 and K_2. Also for $K = \boldsymbol{Q}(\sqrt{d})$ with $723 \leqq |d| < 1000$, we determine $\mathrm{Gal}(K_{ur}/K)$ except for some d. (For table for such fields, see [21]).

Let $K = \boldsymbol{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d < 0$. J. Martinet stated in [12] that if $|d| < 250$, then $K_{ur} = K_1$ except for 7 fields, for which he gave the structure of $\mathrm{Gal}(K_{ur}/K)$. (We note that $\mathrm{Gal}(K_{ur}/K) \cong H_{24}$ for $K = \boldsymbol{Q}(\sqrt{-248})$ in [12] is false). He also stated that this fact is proved by using the methods which J. Masely [13] (and later F. J. van der Linden [18]) used for calculation of class numbers of real abelian number fields of small conductors. They used Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups. In addition to their methods, we use computer for calculation of class numbers of fields of low degrees (we use KANT) and then use class number relations to get class numbers of fields of higher degrees. Results on class field towers [2, 8, 10, 11, and 17] and the knowledge of the 2-groups of orders $\leqq 2^6$ [5] and linear groups over finite fields are also used.

We know that if $|d| \leqq 499$ ($|d| \leqq 2003$ under GRH), then the degree $[K_{ur}:K]$ is finite (see [12]). For these d, we want to determine $\mathrm{Gal}(K_{ur}/K)$. The key fact is that any unramified (finite) extension L of K has the same root discriminant as $K: rd_L = |d_L|^{1/[L:\boldsymbol{Q}]} = rd_K = \sqrt{|d|}$, Thus, if we have $rd_K < B(2N)$, where $B(2N)$ denotes the lower bound for the root discriminants of the totally imaginary number fields of (finite) degrees $\geqq 2N$, then we get $[K_{ur}:K] < N$. We do not know the real values of $B(2N)$ (except for $N \leqq 4$), however, some lower bounds for $B(2N)$ are known. The best known unconditional lower bounds for $B(2N)$ can be found in the tables due to F. Diaz y Diaz [4]. If we assume the truth of GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [14], which are copied in Martinet's expository paper [12]. Let K_l be the top of the class field tower of $K: K = K_0 \subseteqq K_1 \subseteqq K_2 \subseteqq \cdots$ (K_{i+1} is the Hilbert class field of K_i), that is, l is the smallest number with $K_{l+1} = K_l$. If we cannot get $[K_{ur}:K_l] < 60$, which implies $K_{ur} = K_l$, from available lower bounds for $B(2N)$, we need to judge whether K_l has an unramified nonsolvable Galois extension and this is quite difficult. For the fields $\boldsymbol{Q}(\sqrt{-423})$ and $\boldsymbol{Q}(\sqrt{-723})$, we have $h(K_1) = 1$, that is, $l = 1$ and we cannot get $[K_{ur} : K_1] < 60$ from available lower bounds for $B(2N)$ (even under GRH for $\boldsymbol{Q}(\sqrt{-723})$). For $|d| \leqq 420$ ($|d| \leqq 719$ under GRH), we get $[K_{ur} : K_l] < 60$ and our main problem is to determine the degree $[K_l : \boldsymbol{Q}]$. In general, it is difficult to determine $[K_2 : \boldsymbol{Q}]$, because it is very hard to calculate the class number $h(K_1)$ of K_1. (Of course, for K with small $\mathrm{Cl}(K)$, we can calculate $h(K_1)$ with the help of computer). Now let K_g be the genus field of K, that is , the maximal unramified abelian extension of K which is abelian over $\boldsymbol{Q}$. If d is the discriminant of K and $d = d_1 d_2 \cdots d_t$ is the factorization of d into the product of fundamental prime discriminants, then $K_g = \boldsymbol{Q}(\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_t})$, and we have

$$\boldsymbol{Q} \subset K \subseteqq K_g \subseteqq K_1 \subseteqq (K_g)_1 \subseteqq K_2,$$

which implies $[K_2 : \boldsymbol{Q}] = [K_2 : (K_g)_1][(K_g)_1 : \boldsymbol{Q}] = [K_2 : (K_g)_1] h(K_g) [K_g : \boldsymbol{Q}]$. As K_g is a multiquadratic number field, $h(K_g)$ can be calculated by the method in [19], and we may expect that $[K_2 : (K_g)_1]$ is small for fields we consider on the ground of the following proposition (proved in [21]).

Proposition. *Let L be the Hilbert class field of the gunus field K_g of an imaginary abelian number field K. Then for any prime number p with $p \nmid [L : \boldsymbol{Q}]$, the p-class group $\mathrm{Cl}^{(p)}(L)$ of L is trivial or noncyclic.*

As a remarkable fact, for all K with $|d| < 1000$ such that $h(K_g) > h(K)/[K_g : K]$, which is equivalent to $(K_g)_1 \supsetneqq K_1$, we have $K_2 = (K_g)_1$, that is, the second Hilbert class field of K coincides with the Hilbert class field of the genus field K_g of K. For $h(K_g) > h(K)/[K_g : K]$, $h(K)$ must necessarily be even. ($h(K)$ is even if and only if d has (at least) two distinct prime factors), however, for most K, this inequality holds. In fact, if a quadratic subfield $\neq K$ of K_g has class number divisible by an odd prime p, then we have $h(K_g) \geqq p h(K)/[K_g : K]$. Thus, the following question arises naturally:

Question. *Let K be an imaginary abelian number field. Assume that $h(K_g) > h(K)/[K_g : K]$. Then does the equality*

$(*)$ $K_2 = (K_g)_1$

hold? If the answer is not affirmative in general, characterize K for which the equality $(*)$ holds.

The author expects that this problem can be settled group-theoretically and that similar results would also hold for real quadratic number fields.

Except for $\boldsymbol{Q}(\sqrt{-856})$ and $\boldsymbol{Q}(\sqrt{-996})$, we can characterize K (with $|d| < 1000$) for which we can easily get an unramified extension not contained in $(K_g)_1$. If the discriminant d of K is divisible by the discriminant d_E of a quartic number field E, then K has an unramified extension not contained in $(K_g)_1$: The normal closure of E is an S_4-extension of $\boldsymbol{Q}$ unramified at all finite primes over its quadratic subfield $\boldsymbol{Q}(\sqrt{d_E})$. This unramified extension yields an unramified A_4-extension of K_g (by composition), where S_4 (resp. A_4) denotes the symmetric (resp. alternating) group of degree four. Therefore, data for quartic number fields are useful for our study. The fields $\boldsymbol{Q}(\sqrt{-856})$ and $\boldsymbol{Q}(\sqrt{-996})$ are special in the sense that though these fields do not satisfy the condition $d_E \mid d$, we can check that they have an unramified S_4-extension. Thus, $K = \boldsymbol{Q}(\sqrt{d})$ with $|d| < 1000$, can be classified simply as follows:

- $d_E \nmid d$
  - $d \neq -856, -996$
    - $h(K_g) = h(K)/[K_g : K]$
      - $h(K) = 1 \cdots K_{ur} = K$
      - $h(K) > 1 \cdots K_{ur} = K_1$
    - $h(K_g) > h(K)/[K_g : K]$ $\cdots K_{ur} = K_2$
    - (the three cases above: $K_{ur} = (K_g)_1$)
  - $d = -856, -996$ $\cdots$ $l \geqq 3$
- $d_E \mid d$
  - $d = d_E$
    - $d_E = -p$: prime $\cdots K_{ur} = K_3$
    - d_E : composite $\cdots$ $l \geqq 2$
  - $d = d' d_E$ (d': fundamental quadratic discriminant) $\cdots$ $l \geqq 3$
- (the case $d = -856, -996$ and the three cases under $d_E \mid d$: $K_{ur} \supsetneqq (K_g)_1$)

Note that in this classification, there are some possible exceptions. More precisely, for some fields K with $d_E \not| d \neq -856, -996$, we have not succeeded in showing $K_{ur} = (K_g)_1$.

For most K we considered, $K_{ur} = K_l$ is checked. Thus, the following natural question arises: What is the first imaginary quadratic number field having an unramified nonsolvable Galois extension ? (What is the first K with $K_{ur} \neq K_l$?) Recent data for quintic number fields [1 and 16] enable us to give a partial answer:

Proposition. *The field $\boldsymbol{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified A_5-extension which is normal over $\boldsymbol{Q}$ in the sense that none of $\boldsymbol{Q}(\sqrt{d})$ of discriminant d with $0 > d > -1507$ has such an extension. Moreover, such an extension of $\boldsymbol{Q}(\sqrt{-1507})$ is given by the composite field of it with the splitting field of the quintic polynomial $X^5 - 5X^3 + 5X^2 + 24X + 4$, which is an A_5-extension of $\boldsymbol{Q}$.*

We expect that the field $\boldsymbol{Q}(\sqrt{-1507})$ gives the answer to the question above.

For the determination of the structure of $\mathrm{Gal}(K_{ur}/K)$, the results on the 2-class field towers due to H. Kisilevsky [8], F. Lemmermeyer [10 and 11], and E. Benjamin, F. Lemmermeyer, and C. Snyder [2] are very helpful. They give us information on the structure of the Galois group $\mathrm{Gal}(K_2^{(2)}/K)$ of the second Hilbert 2-class field $K_2^{(2)}$ of K over K in many cases.

Now we explain the notations in our table. In the simple expressions of K_1 and K_2, α_i, β_i and γ_i denote any algebraic numbers generating the ith cubic number field of signature (1,1), the ith quartic number field of signature (2,1) with Galois group isomorphic to S_4, and the ith quintic

Table of imaginary quadratic number fields $K = \boldsymbol{Q}(\sqrt{d})$, $|d| \leqq 719$ with $K_{ur} \neq K_1$

$-d$	Cl(K)	K_1	K_2	l	G
115	C_2	$K(\sqrt{5})$	$K_1(\alpha_1)$	2	D_3
120	V_4	$K(\sqrt{-3}, \sqrt{5})$	$K_1(\sqrt{(2\sqrt{2}+\sqrt{5})(2+\sqrt{5})})$	2	Q_8
155	C_4	$K(\sqrt{(-1+5\sqrt{5})/2})$	$K_1(\alpha_2)$	2	Q_{12}
184	C_4	$K(\sqrt{-3+4\sqrt{2}})$	$K_1(\alpha_1)$	2	Q_{12}
195	V_4	$K(\sqrt{-3}, \sqrt{5})$		2	Q_{16}
235	C_2	$K(\sqrt{5})$	$K_1(\gamma_1)$	2	D_5
248	C_8		$K_1(\alpha_2)$	2	I_3^8
255	$C_6 \times C_2$	$K(\sqrt{5}, \sqrt[3]{(9+\sqrt{85})/2})$	$K_1(\sqrt{(5+2\sqrt{-3})(2+\sqrt{5})})$	2	$Q_8 \times C_3$
260	$C_4 \times C_2$	$K(\sqrt{5}, \sqrt{8+\sqrt{65}})$		2	M_{16}
276	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{13+8\sqrt{3}})$	$K_1(\alpha_1)$	2	$Q_{12} \times C_2$
280	V_4	$K(\sqrt{-7}, \sqrt{5})$		2	Q_{16}
283	C_3	$K(\alpha_{31})$	$K_1(\beta_1)$	3	$\widetilde{A}_4$
295	C_8		$K_1(\alpha_4)$	2	I_3^8
299	C_8		$K_1(\alpha_1)$	2	I_3^8
312	V_4	$K(\sqrt{-3}, \sqrt{2})$		2	Q_{16}
331	C_3	$K(\alpha_{36})$	$K_1(\beta_2)$	3	$\widetilde{A}_4$
340	V_4	$K(\sqrt{-1}, \sqrt{5})$		2	SD_{16}
355	C_4	$K(\sqrt{-3+4\sqrt{5}})$		2	Q_{28}
372	V_4	$K(\sqrt{-1}, \sqrt{-3})$	$K_1(\alpha_2)$	2	D_6
376	C_8		$K_1(\gamma_1)$	2	I_5^8
391	C_{14}		$K_1(\alpha_1)$	2	$D_3 \times C_7$
395	C_8		$K_1(\gamma_2)$	2	I_5^8
403	C_2	$K(\sqrt{13})$	$K_1(\alpha_2)$	2	D_3
408	V_4	$K(\sqrt{-3}, \sqrt{2})$	$K_1(\sqrt{-(5+\sqrt{17})/2})$	2	D_4
415	C_{10}	$K(\sqrt{5}, \gamma_{18})$	$K_1(\alpha_6)$	2	$D_3 \times C_5$
420	C_2^3	$K(\sqrt{-1}, \sqrt{-3}, \sqrt{5})$		2	$32\,\Gamma_4 c_3$

Continued (under GRH)

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
435	V_4	$K(\sqrt{-3}, \sqrt{5})$		2	$Q_{16} \ltimes C_3$
440	$C_6 \times C_2$	$K(\sqrt{2}, \sqrt{5}, \alpha_{50})$		2	$Q_{16} \times C_3$
455	$C_{10} \times C_2$	$K(\sqrt{-7}, \sqrt{5}, \gamma_{21})$		2	$Q_8 \times C_5$
472	C_6	$K(\sqrt{2}, \alpha_4)$	$K_1(\alpha_4)$	2	$D_3 \times C_3$
483	V_4	$K(\sqrt{-3}, \sqrt{-7})$	$K_1(\alpha_1)$	2	D_6
491	C_9		$K_1(\beta_3)$	3	$Q_8 \rtimes C_9$
515	C_6	$K(\sqrt{5}, \alpha_{60})$	$K_1(\gamma_3)$	2	$D_5 \times C_3$
520	V_4	$K(\sqrt{-2}, \sqrt{5})$		2	Q_{24}
527	C_{18}		$K_1(\alpha_2)$	2	$D_3 \times C_9$
535	C_{14}		$K_1(\alpha_9)$	2	$D_3 \times C_7$
552	$C_4 \times C_2$	$K(\sqrt{-3}, \sqrt{-1+2\sqrt{6}})$	$K_1(\alpha_1)$	2	$Q_{12} \times C_2$
555	V_4	$K(\sqrt{-3}, \sqrt{5})$		2	Q_{32}
563	C_9		$K_1(\beta_4)$	3	$Q_8 \rtimes C_9$
564	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{1+4\sqrt{3}})$	$K_1(\gamma_1)$	2	$Q_{20} \times C_2$
568	C_4	$K(\sqrt{-1+6\sqrt{2}})$		2	Q_{28}
580	$C_4 \times C_2$	$K(\sqrt{5}, \sqrt{12+\sqrt{145}})$		2	$32\,\Gamma_3 f \ltimes C_3$
595	V_4	$K(\sqrt{-7}, \sqrt{5})$		2	Q_{40}
611	C_{10}	$K(\sqrt{13}, \gamma_{28})$	$K_1(\gamma_1)$	2	$D_5 \times C_5$
632	C_8		$K_1(\gamma_2)$	2	I_5^8
635	C_{10}	$K(\sqrt{5}, \gamma_{31})$	$K_1(\gamma_5)$	2	$D_5 \times C_5$
643	C_3	$K(\alpha_{72})$	$K_1(\beta_5)$	3	$\widetilde{A}_4$
644	$C_8 \times C_2$		$K_1(\alpha_1)$	2	$D_3 \times C_8$
651	$C_4 \times C_2$	$K(\sqrt{-7}, \sqrt{(13+\sqrt{217})/2})$	$K_1(\alpha_2)$	2	$D_3 \times C_4$
655	C_{12}	$K(\sqrt{7+6\sqrt{5}}, \alpha_{75})$	$K_1(\gamma_6)$	2	$Q_{20} \times C_3$
660	C_2^3	$K(\sqrt{-1}, \sqrt{-3}, \sqrt{5})$		2	$64\,\Gamma_{15} f_2$
663	$C_8 \times C_2$			2	$64\,\Gamma_3 p$
664	C_{10}	$K(\sqrt{2}, \gamma_{32})$	$K_1(\alpha_6)$	2	$D_3 \times C_5$
667	C_4	$K(\sqrt{(-13+3\sqrt{29})/2})$	$K_1(\alpha_1)$	2	Q_{12}
680	$C_6 \times C_2$	$K(\sqrt{-2}, \sqrt{5}, \alpha_{80})$		2	$Q_{16} \times C_3$
687	C_{12}	$K(\sqrt{(11+\sqrt{229})/2}, \alpha_{81})$		3	$(A_4 \rtimes C_4) \times C_3$
695	C_{24}		$K_1(\alpha_{13})$	2	$I_3^8 \times C_3$
696	$C_6 \times C_2$	$K(\sqrt{2}, \sqrt[3]{99+13\sqrt{58}})$		2	$Q_{24} \times C_3$
708	V_4	$K(\sqrt{-1}, \sqrt{-3})$	$K_1(\alpha_4)$	2	D_6
715	V_4	$K(\sqrt{-11}, \sqrt{5})$		2	$Q_{16} \ltimes C_5$

number field of signature (1,2) with Galois group isomorphic to D_5, respectively, where we consider that the number fields of each signature and each type (of Galois group of normal closure) are numbered up to conjugacy by absolute values of discriminants. (We do not need to consider nonisomorphic fields with same discriminants).

G denotes the Galois group $\mathrm{Gal}(K_{ur}/K)$. As usual, C_n is the cyclic group of order n, V_4 is the four group, that is, $V_4 = C_2^2 = C_2 \times C_2$, $D_n (n \geqq 3)$ is the dihedral group of order $2n$, $Q_{4n} (n \geqq 2)$ is the generalized quaternion group of order $4n$, and $SD_{8n} (n \geqq 2)$ is the semi-dihedral group of order $8n$. $I_n^{2m} (m \geqq 2, n \geqq 3)$ denotes the group of order $2mm$ given by

$$\langle a, b \mid a^{2m} = b^n = 1, a^{-1}ba = b^{-1} \rangle.$$

$M_{2^n} (n \geqq 4)$ denotes the modular group of order 2^n given by

$$\langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{2^{n-2}+1} \rangle.$$

$\widetilde{A}_4$ is the double cover of $A_4 : \widetilde{A}_4 \cong \mathrm{SL}(2,3)$. For some 2-groups we use designations given in the table by M. Hall and J. K. Senior [5]. We note that T. W. Sag and J. W. Wamsley give minimal presentations for all 2-groups of orders $\leqq 2^6$ [15].

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